Discretization Methods in Fluid Dynamics

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Course 1: Fluid Mechanics and Energy Conversion

Indo-German Winter Academy
OUTLINE

1. Basics of PDE
2. Mathematical Overview of fluid flow
3. Need For Computational Methods
4. Basic Discretization Methods in use
5. Finite Difference Method
6. Fluid Flow Modeling
Partial differential equations (PDEs) are used to formulate and solve problems related to any phenomena that is distributed in **space** and **time**:

- **Heat flow**
- **Fluid flow**
- **propagation of sound**
- **electrodynamics**, 
- **Elasticity**

Most of the fluid and related transport phenomena can be modeled using **PDEs**
Classification of PDEs

\[ A \frac{\partial^2 \phi}{\partial x^2} + B \frac{\partial^2 \phi}{\partial x \partial y} + C \frac{\partial^2 \phi}{\partial y^2} + D \frac{\partial \phi}{\partial x} + E \frac{\partial \phi}{\partial y} + F \phi = G(x, y) \]

If \( A, B, C, D \ldots \) all constants or \( f(x,y) \) \( \quad \longrightarrow \) Linear PDE

If any of \( A, B, C, D \ldots \) contains \( \Phi, \Phi' \) etc \( \quad \longrightarrow \) Non-linear PDE

<table>
<thead>
<tr>
<th>( B^2 - 4AC &lt; 0 )</th>
<th><strong>Elliptic equation</strong></th>
<th>( \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0 )</th>
<th>Irrotational flows steady Heat Transfer</th>
</tr>
</thead>
<tbody>
<tr>
<td>( B^2 - 4AC = 0 )</td>
<td><strong>Parabolic equation</strong></td>
<td>( \frac{\partial \phi}{\partial t} = \alpha \frac{\partial^2 \phi}{\partial x^2} )</td>
<td>Unsteady viscous flows</td>
</tr>
<tr>
<td>( B^2 - 4AC &gt; 0 )</td>
<td><strong>Hyperbolic equation</strong></td>
<td>( \frac{\partial^2 \phi}{\partial t^2} = \beta^2 \frac{\partial^2 \phi}{\partial x^2} )</td>
<td>Vibration problems</td>
</tr>
</tbody>
</table>
Boundary conditions

- **Dirchlet Boundary condition:**
  \[ \phi = \phi_1(r) \]
  Given boundary temperature

- **Neumann Boundary condition:**
  \[ \frac{\partial \phi}{\partial n} = \phi_2(r) \]
  Given boundary heat flux

- **Mixed Boundary condition:**
  \[ a\phi + b\frac{\partial \phi}{\partial n} = \phi_3(r) \]
  Boundary heat flux dependent on boundary temperature
Mathematical description of flows; **Governing Equations of Fluid Dynamics**

- **Equation of continuity**: a differential mass balance

\[
\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z} = -\frac{\partial \rho}{\partial t}
\]

In vector notation: \( \nabla \cdot [\rho \mathbf{u}] = -\frac{\partial \rho}{\partial t} \)

For **constant density** of the fluid: \( \nabla \cdot \mathbf{u} = 0 \)
**Governing Equations of Fluid Dynamics (2)**

**Equation of motion:** \( \frac{\partial}{\partial t} \rho \mathbf{u} + \nabla \cdot (\rho \mathbf{uu}) = -\nabla p - \nabla \cdot \mathbf{\tau} + \rho \mathbf{g} \)

for incompressible, Newtonian Fluids, under laminar flow:

\[
\rho \left( \frac{\partial}{\partial t} \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -\nabla p + \mu \nabla^2 \mathbf{u} + \rho \mathbf{g}
\]

**Navier-Stokes Equation**
The Navier-Stokes equations

\[
\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \frac{\mu}{\rho} \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) + g_x
\]

\[
\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial y} + \frac{\mu}{\rho} \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right) + g_y
\]

\[
\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial z} + \frac{\mu}{\rho} \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right) + g_z
\]

In general, **Analytical Solution Not Possible**

(Non linear PDEs)
Approaches to **Fluid Dynamical Problems**

1. **Simplifications of the governing equations**
   - AFD: Analytical Fluid Dynamic

2. **Experiments on scale models**
   - EFD: Experimental Fluid Dynamics

3. **Discretize governing equations** and solve by computers
   - CFD: Computational Fluid Dynamics

CFD is the simulation of fluids engineering system using modeling and numerical methods.
Need for Discretization methods

- In general, no analytical solution exist for non-linear models. One of the way out is to approximate the solutions using Numerical methods.
- Advent of digital computer and advanced improvements in computer resources.
- Basis for CFD, which can provide unlimited details of results.
  - Substantially more cost effective and more rapid than EFD.
  - Ability to study systems under hazardous conditions.
Various Discretization Techniques

- Finite Difference Method (focused in this lecture)
- Finite Volume Method
- Finite Element Method
Discretization Methods

**Finite Difference Method** (focused in this lecture)

1. Governing equations in differential form \( \rightarrow \) domain with grid \( \rightarrow \) replacing the partial derivatives by approximations in terms of node values of the functions \( \rightarrow \) one algebraic equation per grid node \( \rightarrow \) linear algebraic equation system.
2. Applied to structured grids

**Finite Volume Method**

1. Governing equations in integral form \( \rightarrow \) solution domain is subdivided into a finite number of contiguous control volumes \( \rightarrow \) conservation equation applied to each CV.
2. Computational node locates at the centroid of each CV.
3. Applied to any type of grids, especially complex geometries

**Finite Element Method**

1. Solution domain is subdivided into a finite number OF ELEMENTS, the governing equation is solved for each element and then overall solution is obtained by “assembly”.
2. Equations are multiplied by a weight function before integrated over the entire domain.
Basics aspects of Discretization methods

► **Consistency**

A discretization method is said to be consistent if it can be shown that the difference between PDEs and its finite difference (FDE) vanishes as the mesh is refined.

**Truncation error (TE):** Difference between the discretized equation and the exact one

\[ \lim_{\text{mesh}\to 0} (\text{TE}) \to 0 \]

- \( O(\Delta x); O(\Delta t) \) ✔
- \( O(\Delta t/\Delta x) \) ✗

► **Convergence:** solution of the discretized equations tends to the exact solution of the differential equation as the grid spacing tends to zero.
**Basics aspects of Discretization methods**

**Stability**: Differencing Method should not magnify the errors that appear in the course of numerical solution process

\[
\frac{\varepsilon_i^{n+1}}{\varepsilon_i^n} \leq 1
\]

**Conservation:**

1. The numerical scheme should on both local and global basis respect the conservation laws.
2. Automatically satisfied for control volume method, either individual control volume or the whole domain.
3. Errors due to non-conservation are in most cases appreciable only on relatively coarse grids, but hard to estimate quantitatively.
**Finite Difference Method**

Replacing the derivatives of governing PDE with finite, algebraic differences quotients. It involves following steps:

- **Grid generation** → defining geometric domain
- **Discretization** of governing equation using **Taylor series approximation**.
- **Solution of simultaneous algebraic equations**.

Discrete grid points
Finite Difference Method: grids

Grids

- Structured grid
  - all nodes have the same number of elements around it
  - only for simple domains
- Unstructured grid
  - for all geometries
  - irregular data structure
Finite Difference, approximation of the **first derivative**

**Taylor Series Expansion:** Any continuous differentiable function, in the vicinity of \( x_i \), can be expressed as a Taylor series:

\[
\Phi(x) = \Phi(x_i) + (x - x_i) \left( \frac{\partial \Phi}{\partial x} \right)_i + \frac{(x - x_i)^2}{2!} \left( \frac{\partial^2 \Phi}{\partial x^2} \right)_i + \frac{(x - x_i)^3}{3!} \left( \frac{\partial^3 \Phi}{\partial x^3} \right)_i + \ldots + \frac{(x - x_i)^n}{n!} \left( \frac{\partial^n \Phi}{\partial x^n} \right)_i + H
\]

\[
\left( \frac{\partial \Phi}{\partial x} \right)_i = \frac{\Phi_{i+1} - \Phi_i}{x_{i+1} - x_i}
\]

\[
\left( \frac{\partial \Phi}{\partial x} \right)_{i,j} = \left\{ \frac{\Phi_{i+1,j} - \Phi_{i,j}}{\Delta x} \right\} - \left\{ \frac{\partial^2 \Phi}{\partial x^2} \right\}_{i,j} \frac{\Delta x}{2} + \left\{ \frac{\partial^3 \Phi}{\partial x^3} \right\}_{i,j} \frac{\Delta x^2}{6} + \ldots
\]

**Finite Difference Quotient**

\[
\left( \frac{\partial \Phi}{\partial x} \right)_{i,j} = \left\{ \frac{\Phi_{i+1,j} - \Phi_{i,j}}{\Delta x} \right\} + O(\Delta x)
\]

**T.E (Truncation Error)**

\[
O(\Delta x)
\]

**I Order Accurate**

\[
\Delta x \quad (+) \quad (i,j) \quad (i+1,j)
\]
By writing Taylor series at different nodes, $x_{i-1}$, $x_{i+1}$, or both $x_{i-1}$ and $x_{i+1}$, we can have:

<table>
<thead>
<tr>
<th>Scheme</th>
<th>Order of Accuracy</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Forward Difference Scheme (FDS)</strong></td>
<td>$O(\Delta x)$</td>
</tr>
<tr>
<td>$\left( \frac{\partial \Phi}{\partial x} \right)<em>i \approx \frac{\Phi</em>{i+1} - \Phi_i}{x_{i+1} - x_i}$</td>
<td></td>
</tr>
<tr>
<td><strong>Backward Difference Scheme (BDS)</strong></td>
<td>$O(\Delta x)$</td>
</tr>
<tr>
<td>$\left( \frac{\partial \Phi}{\partial x} \right)<em>i \approx \frac{\Phi_i - \Phi</em>{i-1}}{x_i - x_{i-1}}$</td>
<td></td>
</tr>
<tr>
<td><strong>Central Difference Scheme (CDS)</strong></td>
<td>$O(\Delta x)^2$</td>
</tr>
<tr>
<td>$\left( \frac{\partial \Phi}{\partial x} \right)<em>i \approx \frac{\Phi</em>{i+1} - \Phi_{i-1}}{x_{i+1} - x_{i-1}}$</td>
<td></td>
</tr>
</tbody>
</table>
Finite Difference, approximation of the second derivative

Geometrically, the second derivative is the slope of the line tangent to the curve representing the first derivative.

\[
\left( \frac{\partial^2 \Phi}{\partial x^2} \right)_i \approx \frac{\left( \frac{\partial \Phi}{\partial x} \right)_{i+1} - \left( \frac{\partial \Phi}{\partial x} \right)_i}{x_{i+1} - x_i}
\]

Estimate the outer derivative by FDS, and estimate the inner derivatives using BDS, we get

\[
\left( \frac{\partial^2 \Phi}{\partial x^2} \right)_i \approx \frac{\Phi_{i+1}(x_i - x_{i-1}) + \Phi_{i-1}(x_{i+1} - x_i) - \Phi_i(x_{i+1} - x_{i-1})}{(x_{i+1} - x_i)^2(x_i - x_{i-1})}
\]

For equidistant spacing of the points:

\[
\left( \frac{\partial^2 \Phi}{\partial x^2} \right)_i \approx \frac{\Phi_{i+1} - 2\Phi_i + \Phi_{i-1}}{(\Delta x)^2}
\]

Higher-order approximations for the second derivative can be derived by including more data points, such as \(x_{i-2}\), and \(x_{i+2}\), even \(x_{i-3}\), and \(x_{i+3}\)
Discretization using Explicit Method

Consider unsteady viscous flow

\[
\frac{\partial \Phi}{\partial t} = \alpha \frac{\partial^2 \Phi}{\partial x^2}
\]

\[
\frac{\Phi_{i}^{n+1} - \Phi_{i}^{n}}{\Delta t} = \alpha \frac{\Phi_{i+1}^{n+1} - 2\Phi_{i}^{n} + \Phi_{i-1}^{n}}{(\Delta x)^2}
\]

The Only unkown: \( \Phi_{i}^{n+1} \)

**Explicit method**, values at time n+1 computed from values at time n

**Advantages**: solution algorithm is simple i.e.
- direct computation without solving system of algebraic equation
- few number of operations per time step

Requires many time steps to carry out the calculations over a given time interval
Implicit Method
Crank-Nicholson Method

\[
\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}
\]

\[
\frac{u_{i}^{n+1} - u_{i}^{n}}{\Delta t} = \alpha \frac{u_{i+1}^{n} + u_{i+1}^{n+1} - 2u_{i}^{n} - 2u_{i}^{n+1} + u_{i-1}^{n} + u_{i-1}^{n+1}}{2(\Delta x)^2}
\]

- values at time \( n+1 \) computed using the unknown values at time \( n+1 \)
- Second order accurate in time and space \( O(\Delta t)^2, O(\Delta x)^2 \)
- to be solved simultaneously at all the grid points as a system of algebraic equations
- Unconditionally stable

rearranging the above equation as follows….
\[ u_{i+1}^n - u_i^n = \frac{r}{2} [u_{i+1}^{n+1} + u_{i+1}^n - 2u_{i+1}^n - 2u_i^n + u_{i-1}^{n+1} + u_{i-1}^n] \]

where \( r = \alpha(\Delta t)/(\Delta x)^2 \) or

\[-ru_{i-1}^{n+1} + (2 + 2r)u_i^{n+1} - ru_{i+1}^n = ru_{i-1}^n + (2 - 2r)u_i^n + ru_{i+1}^n \]

or

\[-u_{i-1}^{n+1} + \left(\frac{2 + 2r}{r}\right)u_{i}^{n+1} - u_{i+1}^{n+1} = u_{i-1}^n + \left(\frac{2 - 2r}{r}\right)u_i^n + ru_{i+1}^n \]

has to be applied at all grid points, i.e., from \( i = 1 \) to \( i = k + 1 \). A system of algebraic equations will result

at \( i = 2 \) \quad -A + B(1)u_2^{n+1} - u_3^{n+1} = C(1) \\

at \( i = 3 \) \quad -u_2^{n+1} + B(2)u_3^{n+1} - u_4^{n+1} = C(2) \\

at \( i = 4 \) \quad -u_3^{n+1} + B(3)u_4^{n+1} - u_5^{n+1} = C(3) \\

\vdots \quad \vdots \\

at \( i = k \) \quad -u_{k-1}^{n+1} + B(k-1)u_k^{n+1} - D = C(k-1) \]
Finally the equations will be of the form:

\[
\begin{bmatrix}
B(1) & -1 & 0 & 0 & \ldots & 0 \\
-1 & B(2) & -1 & 0 & \ldots & 0 \\
0 & -1 & B(3) & -1 & \ldots & 0 \\
\vdots \\
0 & 0 & 0 & \ldots & -1 & B(k-1)
\end{bmatrix}
\begin{bmatrix}
u_2^{n+1} \\
u_3^{n+1} \\
u_4^{n+1} \\
\vdots \\
u_k^{n+1}
\end{bmatrix}
= 
\begin{bmatrix}
(C(1) + A)^n \\
C(2)^n \\
C(3)^n \\
\vdots \\
(C(k-1) + D)^n
\end{bmatrix}
\]

\[Ax = C\]

This generates a Tridiagonal Matrix system which can be solved using Thomas algorithm.

However, for higher dimensional flows, matrix system obtained are much more complex and require substantially more computer time.
Alternating Direction Implicit Method (ADI) for 2-D Flows

\[
\frac{\partial u}{\partial t} = \alpha \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)
\]

Each time increment is executed into two steps:

First step

\[
\frac{u_{i,j}^{n+1/2} - u_{i,j}^{n}}{\Delta t/2} = \alpha \left( \frac{u_{i+1,j}^{n} - 2u_{i,j}^{n} + u_{i-1,j}^{n}}{(\Delta x)^2} \right) + \frac{u_{i,j+1}^{n+1/2} - 2u_{i,j}^{n+1/2} + u_{i,j-1}^{n+1/2}}{(\Delta y)^2}
\]

Second step

\[
\frac{u_{i,j}^{n+1} - u_{i,j}^{n+1/2}}{\Delta t/2} = \alpha \left( \frac{u_{i+1,j}^{n+1} - 2u_{i,j}^{n+1} + u_{i-1,j}^{n+1}}{(\Delta x)^2} \right) + \frac{u_{i,j+1}^{n+1/2} - 2u_{i,j}^{n+1/2} + u_{i,j-1}^{n+1/2}}{(\Delta y)^2}
\]
ADI method results in **Tridiagonal Equations** (for 2-d flow systems) if it is applied along the dimension that is implicit. Thus on first step it is applied along Y axis and on second step along on X axis.
Implicit method

- Advantages:
  - Stability can be maintained over larger time steps

- Disadvantages
  - More computation time per time step, every time step requires solving a system of equations.
  - Truncation error is often large, since larger time steps are employed.
  - Overall, a more involved procedure
Error and Stability analysis

- Numerical solution is influenced by following:
  - Discretization errors or Truncation errors:
    Analytical Solution (A) – exact Finite Difference Solution (D)
    \[ = A - D \]
  - Computational Round-Off Error (\( \varepsilon \))
    \[ \varepsilon = \text{Numerical Solution (N)} - \text{exact Finite Difference Solution (D)} \]
    \[ N = D + \varepsilon \]
    By definition, N,D follow the Discretized Equation.
    Thus, \( \varepsilon \) should also follow the equation:

    \[
    \frac{\varepsilon^{n+1}_i - \varepsilon^n_i}{\Delta t} = \alpha \frac{\varepsilon^{n+1}_{i+1} - 2\varepsilon^n_i + \varepsilon^n_{i-1}}{(\Delta x)^2}
    \]
Errors and Stability Analysis (Cont.)

- Substituting the value of $\varepsilon_m(x,t)$ in the F.D.E.,

$$e^{a \Delta t} = 1 + \frac{\alpha \Delta t}{\Delta x^2} \left( e^{ik_m \Delta x} + e^{-ik_m \Delta x} - 2 \right)$$

$$e^{a \Delta t} = 1 + \frac{2\alpha \Delta t}{\Delta x^2} \left( \cos(k_m \Delta x) - 1 \right)$$

$$= 1 - \frac{4\alpha \Delta t}{\Delta x^2} \sin^2 \left( \frac{k_m \Delta x}{2} \right)$$

$$\therefore \frac{\varepsilon_i^{n+1}}{\varepsilon_i^n} = \frac{e^{a(t+\Delta t)} e^{ik_m x}}{e^{a t} e^{ik_m x}} = e^{a \Delta t}$$
**Stability**: Differencing Method should not magnify the errors that appear in the course off numerical solution process

\[
\left| \frac{\varepsilon_i^{n+1}}{\varepsilon_i^n} \right| \leq 1
\]

**Von Neumann Stability Analysis**: assuming error distribution follows Fourier series in space domain and exponential in time

\[
\varepsilon(x, t) = \varepsilon^a \sum_m \varepsilon^{ik_m x}
\]

Putting an individual term in FD equation and using the above stability criteria

\[
\frac{\alpha(\Delta t)}{(\Delta x)^2} \leq \frac{1}{2}
\]
Fluid Flow Modeling Using Finite Difference Method

- **Nonlinear Fluid Flow equations involve:**
  - time dependent multiple variables
  - Convective terms
  - Diffusive terms

- **Sample fluid flow equation : Burger’s Equation**

\[
\frac{\partial u}{\partial t} + u \frac{\partial \zeta}{\partial x} = \nu \left( \frac{\partial^2 \zeta}{\partial x^2} \right)
\]

- **Inviscid Burger’s Equation**

\[
\frac{\partial u}{\partial t} + u \frac{\partial \zeta}{\partial x} = 0
\]
Properties of Fluid Flow Equations

Effect of **Finite Difference** method on properties of fluid flow equations

- **Conservative Property:** To maintain the integral conservation relation of continuum in finite difference representation. It depends on the form of continuum equation being discretized.

  Conservative form: \[ \frac{\partial \zeta}{\partial t} = -\frac{\partial}{\partial x}(u\zeta) \]

  Non-conservative form: \[ \frac{\partial \zeta}{\partial t} = -u \frac{\partial \zeta}{\partial x} \]
Integrating the
CONSERVATIVE FORM

\[
\int \frac{\zeta_{i+1}^n - \zeta_i^n}{\Delta t} \Delta x = \alpha \int \frac{(u\zeta)^n_{i+1} - (u\zeta)^n_{i-1}}{2\Delta x} \Delta x
\]

\[
\sum_{i=I_1}^{I_2} \frac{\zeta_{i+1}^n - \zeta_i^n}{\Delta t} \Delta x = \frac{\sum_{i=I_1}^{I_2} (u_i^n \zeta_{i+1}^n - u_i^n \zeta_i^n)}{2}
\]

\[
= (u\zeta)_{I_1 - 1/2} - (u\zeta)_{I_2 + 1/2}
\]
Week Conservative Form in Fluid Flow

- Conservative form of **advection part** is of prime importance for fluid flow modeling. Navier Stokes in week conservative form:

\[
\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} + \frac{\partial uv}{\partial y} + \frac{\partial uw}{\partial z} = \frac{1}{\rho} \frac{\partial p}{\partial x} + \frac{\mu}{\rho} \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) + g_x
\]

\[
\frac{\partial v}{\partial t} + \frac{\partial u}{\partial x} + \frac{\partial v^2}{\partial y} + \frac{\partial vw}{\partial z} = \frac{1}{\rho} \frac{\partial p}{\partial y} + \frac{\mu}{\rho} \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right) + g_y
\]

\[
\frac{\partial w}{\partial t} + \frac{\partial uw}{\partial x} + \frac{\partial vw}{\partial y} + \frac{\partial w^2}{\partial z} = \frac{1}{\rho} \frac{\partial p}{\partial z} + \frac{\mu}{\rho} \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right) + g_z
\]
Discretizing Fluid Flow equations using Upwind Scheme

- Upwind Scheme of discretization is necessary for convection dominated flows
  \[
  \frac{\partial \xi}{\partial t} = -u \frac{\partial \xi}{\partial x}
  \]

- **Backward Differencing Scheme** for \( u > 0 \)
  \[
  \frac{\xi_{i+1} - \xi_i}{\Delta t} = -\frac{u\xi_{i+1} - u\xi_i}{\Delta x}
  \]

- **Forward Differencing Scheme** for \( u < 0 \)
  \[
  \frac{\xi_{i+1} - \xi_i}{\Delta t} = \frac{u\xi_{i+1} - u\xi_i}{\Delta x}
  \]
Why use Upwind scheme

- Upwind Scheme of discretization is necessary for **convection dominated flows** for obtaining **numerically stable results**

- Central Difference Scheme applied to highly convective flows give unconditionally Unstable Results

\[
\begin{align*}
\text{CDS: } & \quad \frac{\zeta_{i+1}^n - \zeta_i^n}{\Delta t} + \frac{u \zeta_{i+1}^n - u \zeta_i^n}{2\Delta x} = 0 \\
\text{FDS: } & \quad \frac{\zeta_{i+1}^n - \zeta_i^n}{\Delta t} + \frac{u \zeta_{i+1}^n - u \zeta_i^n}{\Delta x} = 0, \quad u > 0
\end{align*}
\]

Violates Von Neumann Stability Criteria
Why use Upwind scheme

For unsteady, convection-diffusion eqn.

\[
\frac{\xi_i^{n+1} - \xi_i^n}{\Delta t} + \frac{u \xi_i^{n+1} - u \xi_i^{n-1}}{2\Delta x} = \nu \frac{\xi_i^{n+1} - 2\xi_i^n + \xi_i^{n-1}}{\Delta x^2}
\]

CDS:

\[
\frac{\partial \xi}{\partial t} + u \xi_x = \left( \nu - \left( \frac{u^2}{2} \frac{\Delta t}{\Delta x} \right) \right) \xi_{xx}
\]

\[
\left( \nu - \left( \frac{u^2 \Delta t}{2} \right) \right) \geq 0
\]

\[
\nu \left( 1 - \frac{1}{2} \frac{u \Delta t}{\Delta x} \frac{u \Delta x}{\nu} \right) \geq 0
\]

\[
\nu \left( 1 - \frac{1}{2} C \cdot \text{Re} \right) \geq 0
\]

\[
\text{Re} \leq \frac{2}{C}
\]

\[
\text{Re}_x = \left( \frac{u \Delta x}{\nu} \right)
\]

CELL Reynold’s No.

Upwind Scheme and Transportive Property

\[ \frac{\partial \zeta}{\partial t} = -\frac{\partial}{\partial x} (u \zeta) \]

using upwind scheme \((u > 0) \Rightarrow \frac{\zeta_{i+1}^{n} - \zeta_{i}^{n}}{\Delta t} = -\frac{u \zeta_{i}^{n} - u \zeta_{i-1}^{n}}{\Delta x} \)

let a perturbation on \(\delta\) in \(\zeta\) occurs at \(m^{th}\) location

• Then for at a downstream location \((m + 1)\)

\[ \frac{\zeta_{m+1}^{n+1} - \zeta_{m+1}^{n}}{\Delta t} = -\frac{0 - u\delta}{\Delta x} = \frac{u\delta}{\Delta x} \]

which follows the rationale for the transportive property

• At \((m)\) location

\[ \frac{\zeta_{m}^{n+1} - \zeta_{m}^{n}}{\Delta t} = -\frac{u\delta - 0}{\Delta x} = -\frac{u\delta}{\Delta x} \]

perturbation being transported out of the region

• At \((m - 1)\) location

\[ \frac{\zeta_{m-1}^{n+1} - \zeta_{m-1}^{n}}{\Delta t} = -\frac{0 - 0}{\Delta x} = 0 \]

no perturbation carried upstream
Problems of Using Forward Time Central Space method (FTCS)

\[
\frac{\zeta_{i}^{n+1} - \zeta_{i}^{n}}{\Delta t} = -\frac{u \zeta_{i+1}^{n} - u \zeta_{i-1}^{n}}{2\Delta x}
\]

At (m - 1) location

\[
\frac{\zeta_{m-1}^{n+1} - \zeta_{m-1}^{n}}{\Delta t} = -\frac{u \delta - 0}{2\Delta x} = -\frac{u \delta}{2\Delta x} \neq 0
\]

Which indicates that Transportive Property is **violated**

Hence, only upwind method maintains unidirectional flow of information.
Upwind Method and Artificial Viscosity (1/2)

\[
\frac{\partial u}{\partial t} + u \frac{\partial \zeta}{\partial x} = \nu \left( \frac{\partial^2 \zeta}{\partial x^2} \right)
\]

\[
\frac{\zeta_i^{n+1} - \zeta_i^n}{\Delta t} = \frac{u\zeta_i^n - u\zeta_{i-1}^n}{\Delta x}
\]

\[
\frac{\zeta_i^{n+1} - \zeta_i^n}{\Delta t} = \frac{u\zeta_{i+1}^n - u\zeta_i^n}{\Delta x}
\]

From Taylor series expansion, we can write

\[
\zeta_i^{n+1} = \zeta_i^n + \Delta t \frac{\partial \zeta}{\partial t} \bigg|_i^n + \frac{(\Delta t)^2}{2} \frac{\partial^2 \zeta}{\partial t^2} \bigg|_i^n + \cdots
\]

\[
\zeta_{i\pm 1}^{n+1} = \zeta_i^n \pm \Delta x \frac{\partial \zeta}{\partial x} \bigg|_i^n + \frac{(\Delta x)^2}{2} \frac{\partial^2 \zeta}{\partial x^2} \bigg|_i^n \pm \cdots
\]
Upwind Method and Artificial Viscosity (2/2)

\[
\frac{\partial \zeta}{\partial t} = -u \frac{\partial \zeta}{\partial x} + \frac{1}{2} \left[ u \Delta x \left( 1 - \frac{u \Delta t}{\Delta x} \right) \right] \frac{\partial^2 \zeta}{\partial x^2} + \nu \frac{\partial^2 \zeta}{\partial x^2} + \mathcal{O}(\Delta x)^2
\]

which may be rewritten as

\[
\frac{\partial \zeta}{\partial t} = -u \frac{\partial \zeta}{\partial x} + \nu \frac{\partial^2 \zeta}{\partial x^2} + \nu_e \frac{\partial^2 \zeta}{\partial x^2} + \text{higher-order terms}
\]

where

\[
\nu_e = \frac{1}{2} \left[ u \Delta x \left( 1 - C \right) \right], \quad C \ (\text{Courant number}) = \frac{u \Delta t}{\Delta x} < 1
\]

For algorithm's stability, artificial viscosity will necessarily be present

\[
\text{Source of inaccuracy} \quad \nu_e > 0
\]
Upwind Differencing Method
Advantages and Limitations

- Upwind effect necessary to maintain transportive property of flow equations

- Although space centered differences (CDS) are more accurate than upwind scheme (as indicated by Taylor Series expansion), the whole system is more realistically modeled using Upwind Differencing Scheme.

- Leads to Artificial viscosity or “false diffusion” which is a cause of inaccuracy. 
  \[ v_e > 0 \]

Plausible Improvement: Higher order Upwind Method
Second Upwind Differencing Scheme

Let us consider

\[
\frac{\partial (u\zeta)}{\partial x}\bigg|_{i,j} = \frac{u_R \zeta_R - u_L \zeta_L}{\Delta x}
\]

where

\[
u_R = \frac{u_{i,j} + u_{i+1,j}}{2}; \quad u_L = \frac{u_{i,j} + u_{i-1,j}}{2}
\]

for \( u_R > 0 \) and \( u_L > 0 \), we get

\[
\frac{\partial (u\zeta)}{\partial x} = \frac{1}{\Delta x} \left[ \left( \frac{u_{i,j} + u_{i+1,j}}{2} \right) \zeta_{i,j} - \left( \frac{u_{i,j} + u_{i-1,j}}{2} \right) \zeta_{i-1,j} \right]
\]
Considering momentum equation  \( \zeta = u \)

\[
\left. \frac{\partial u_i^2}{\partial x} \right|_{i,j} = \frac{1}{\Delta x} \left[ \left( \frac{u_{i+1,j} + u_{i,j}}{2} \right) u_{i,j} - \left( \frac{u_{i,j} + u_{i-1,j}}{2} \right) u_{i-1,j} \right] 
\]

\[
= \frac{1}{4\Delta x} \left[ \left( u_{i,j} + u_{i+1,j} \right)^2 + \left( u_{i,j}^2 - u_{i+1,j}^2 \right) \right. \\
- \left( u_{i-1,j} + u_{i,j} \right)^2 - \left( u_{i-1,j}^2 - u_{i,j}^2 \right) \left. \right] 
\]

Here a factor \( \eta \) is introduced which expresses a weighted average of central and upwind differencing

\[
\left. \frac{\partial u_i^2}{\partial x} \right|_{i,j} = \frac{1}{4\Delta x} \left[ \left( u_{i,j} + u_{i+1,j} \right)^2 + \eta (u_{i,j} + u_{i+1,j}) \right. \\
- \left( u_{i-1,j} + u_{i,j} \right)^2 - \left( u_{i-1,j}^2 - u_{i,j}^2 \right) \left. \right] 
\]
\[
\frac{\partial u^2}{\partial x} - \left| \frac{1}{4\Delta x} \left[ (u_{i,j} + u_{i+1,j})^2 + \eta (u_{i,j} + u_{i+1,j})(u_{i,j} - u_{i+1,j}) \right. \right. \\
\left. \left. - (u_{i-1,j} + u_{i,j})^2 - \eta (u_{i-1,j} + u_{i,j})(u_{i-1,j} - u_{i,j}) \right] \right.
\]

Here 0<\(\eta<1\)

- For \(\eta=0\), above equation becomes centered in space
- For \(\eta=1\), above equation follows full upwind
- Hence, \(\eta\) brings about an upwind bias in the difference quotient
- Accuracy of the differencing scheme can be adjusted using \(\eta\) value

Second upwind formulation possesses both the conservative and transportive property
Summary

- Finite Difference methods discretizes the differential form of governing equation using Taylor Series expansion.
- The partial derivatives of governing PDE are replaced with finite, algebraic differences quotients at the corresponding nodes.
- Leads to linear algebraic equation system, with one algebraic equation per grid node.
- Best Suited for Structured Grids only.
- Flow equation properties can be maintained using simple schemes like upwind differencing.

Discretization methods provide more cost effective and more rapid approach for solving Fluid Flow problems as compared to experimental methods.
Summary

Finite Difference Approach

FD Approaches Can Be Divided Into Various Categories Depending Upon Type of PDE Being Solved:

1. Elliptic PDE: Algebraic System, Gauss Siedel
2. One-dimensional Parabolic PDE
   A) Explicit
   B) Implicit
3. Two-dimensional Parabolic PDE
   ADI Method
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References


- **Anderson.J.;** “Computational Fluid Dynamics”
Thank you!