Similarity solution of the Boundary layer equation

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Outline

1. Boundary Layer equation
   - Boundary Layer
   - Order of Magnitude analysis

2. Similarity solution
   - Scaling

3. Falkner Skan Transformation
   - Boundary conditions
   - Method of solution

4. Thermal Boundary layers
   - Thermal Boundary layer equation
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Boundary Layer

- Classical problem: Flow past a flat plate at zero angle of attack
- Prandtl's hypothesis
- Characteristic length scales
  - $x, \Delta x \sim L$
  - $y, \Delta y \sim \delta$
  - $u \sim V_\infty$

Figure: A typical boundary layer
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   - Thermal Boundary layer equation
Order of Magnitude analysis

- Continuity equation - Incompressible flow

\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \implies v \sim \frac{V_\infty \delta}{L} \]

- U momentum equation

\[ u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + \frac{1}{\rho} \frac{\partial P}{\partial x} = \nu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \]

\[ \begin{align*}
  u \frac{\partial u}{\partial x} & \sim \frac{V_\infty^2}{L} \\
  v \frac{\partial u}{\partial y} & \sim \frac{V_\infty^2}{L} \\
  \frac{1}{\rho} \frac{\partial P}{\partial x} & \sim ? \\
  \nu \frac{\partial^2 u}{\partial x^2} & \sim \frac{1}{Re^{3/2}} \frac{V_\infty^2}{L} \\
  \nu \frac{\partial^2 u}{\partial y^2} & \sim \nu \frac{V_\infty}{\delta^2}
\end{align*} \]
Order of Magnitude analysis (contd.)

\[
\text{Is } \nu \frac{V_\infty}{\delta^2} \sim \frac{V_\infty^2}{L} \text{? }
\]

If \( \nu \frac{V_\infty}{\delta^2} \ll \frac{V_\infty^2}{L} \rightarrow \text{Euler equation} \)

If \( \nu \frac{V_\infty}{\delta^2} \gg \frac{V_\infty^2}{L} \rightarrow \text{Hele Shaw flow} \)

In the B.L, we have \( \nu \frac{V_\infty}{\delta^2} \sim \frac{V_\infty^2}{L} \Rightarrow \frac{\delta}{L} \sim \frac{1}{\sqrt{Re}} \)
Order of Magnitude analysis (contd.)

- Hence the u-mom equation reduces to

\[ u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + \frac{1}{\rho} \frac{\partial P}{\partial x} = \nu \frac{\partial^2 u}{\partial y^2} \]

- v-mom equation

\[ u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + \frac{1}{\rho} \frac{\partial P}{\partial y} = \nu \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) \]

- Compared to the u-mom equation, all the terms are small.

Like \( \frac{V^2 \delta}{L^2} \), \( \frac{\nu V^2}{L^2} \) etc.
Order of Magnitude analysis (contd.)

- What v-mom equation tells us is $\frac{\partial P}{\partial y} = 0$ in the B.L
- External pressure $\rightarrow$ imposed on the surface.
- Analogy with ant crawling on a surface
- In the external flow field (inviscid, potential flow)

$$Pe + \frac{1}{2} \rho v_e^2 = const \implies \frac{1}{\rho} \frac{dP_e}{dx} = U_e \frac{dU_e}{dx}$$

- B.L responds to external flow through the imposed pressure
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   - Thermal Boundary layer equation
The velocity profiles $U(x_1)$ and $U(x_2)$ differ only by a scale factor

- $U_\infty \rightarrow$ Obvious scale factor for $u$
- New co-ordinate $\rightarrow$ Dimensionless $\rightarrow$ scale factor $g(x)$
- $u = u(U_\infty, \nu, x, y) \rightarrow$ 5 variables in 2 dimensions
- Dimensional analysis $\rightarrow$ dimensionless quantity in 3 variables
Scaling

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Dimensional analysis $\rightarrow$ dimensionless quantity in 3 variables
Scaling (contd.)

- Need to retain \( y \) as it is. Variable to be scaled. Can’t use ‘\( u \)’ as well.

- Hence

\[
\frac{u}{U_\infty} = F \left( \frac{y}{\delta} \right) = F \left( \frac{y}{\sqrt{\nu x / U_\infty}} \right)
\]

\[
\delta = \sqrt{\frac{\nu x}{U_\infty}} \quad \eta = \frac{y}{\sqrt{\nu x / U_\infty}}
\]
Scaling (contd.)

Getting the stream function

\[ \psi = \int u dy = \int U_\infty F(\eta) \sqrt{\frac{\nu x}{U_\infty}} \, d\eta = \sqrt{U_\infty \nu x} \int F(\eta) \, d\eta \]

\[ \psi = \sqrt{U_\infty \nu x} f(\eta) + C \]
**Falkner Skan Transformation**

\[
\begin{align*}
    u &= \frac{\partial \psi}{\partial y} = \frac{\partial \psi}{\partial \eta} \frac{\partial \eta}{\partial y} = u_e f' \\
    v &= -\frac{\partial \psi}{\partial x} = \frac{f'}{2x} u_e y - \frac{f' y}{2} \frac{d u_e}{d x} - \frac{1}{2} \sqrt{\frac{\nu u_e}{x}} \, f \\
\end{align*}
\]

U mom equation becomes (an ODE !!)

\[
\frac{d u_e}{d x} \left( f'^2 - 1 \right) - \frac{f'''}{2} \left( \frac{u_e}{x} + \frac{d u_e}{d x} \right) = \frac{u_e}{x} f'''
\]

This equation gets greatly simplified if \( \frac{d u_e}{d x} = m \frac{u_e}{x} \)
Falkner Skan Transformation

\[ u = \frac{\partial \psi}{\partial y} = \frac{\partial \psi}{\partial \eta} \frac{\partial \eta}{\partial y} = u_e f' \]

\[ v = -\frac{\partial \psi}{\partial x} = \frac{f'}{2} u_e y - \frac{f' y}{2} \frac{du_e}{dx} - \frac{1}{2} \sqrt{\nu u_e x} f \]

U mom equation becomes (an ODE !!)

\[ \frac{du_e}{dx} (f'^2 - 1) - \frac{ff'''}{2} \left( \frac{u_e}{x} + \frac{du_e}{dx} \right) = \frac{u_e}{x} f''' \]

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U mom equation becomes (an ODE !!)

\[ \frac{du_{e}}{dx} \left( f'^{2} - 1 \right) - \frac{ff'''}{2} \left( \frac{u_{e}}{x} + \frac{du_{e}}{dx} \right) = \frac{u_{e}}{x} f''' \]

This equation gets greatly simplified if \( \frac{du_{e}}{dx} = m \frac{u_{e}}{x} \)
Falkner Skan Transformation (contd.)

- \( \frac{du_e}{dx} = m \frac{u_e}{x} \rightarrow \text{Actual flows?} \)
- \( u_e = Ax^m \rightarrow \text{Flow past a wedge of included angle } \beta \)

Figure: Wedge flow configuration
Blasius and Hiemenz flow

- Case $m = 0 \rightarrow$ Blasius flow
  \[ 2f_{\eta\eta\eta} + ff_{\eta\eta} = 0 \]

- Case $m = \pi \rightarrow$ Stagnation point flow (Hiemenz flow)
  \[ f_{\eta\eta\eta} + ff_{\eta\eta} - f_{\eta}^2 + 1 = 0 \]
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Boundary conditions

- No slip wall \( u = 0 \) and Rigid wall \( v = 0 \) @ \( y = 0 \)
  \[ f'(\eta) = 0 = f(\eta) \] @ \( \eta = 0 \)

- Integration with Free Stream velocity @ \( y = \infty \), \( u = U_\infty \)
  \[ @ \eta = \infty, f'(\eta) = 1 \]
Boundary conditions

- No slip wall \( u = 0 \) and Rigid wall \( v = 0 \) @ \( y = 0 \)
  \( \Rightarrow f'(\eta) = 0 = f(\eta) \) @ \( \eta = 0 \)

- Integration with Free Stream velocity @ \( y = \infty \), \( u = U_\infty \)
  \( \Rightarrow @ \ \eta = \infty \), \( f'(\eta) = 1 \)
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Series solution

\[ f(\eta) = A_0 + A_1 \eta + \frac{A_2}{2!} \eta^2 + \frac{A_3}{3!} \eta^3 + \frac{A_4}{4!} \eta^4 + \ldots \]

- Resembles Frobenius method of solution
- Local solution - Series expanded about \( \eta = 0 \)
- \( \eta = 0 \) → Region of interest → \( C_f \) → Skin friction coefficient
Numerical solution

- Lets take Blasius eqn $2f''' + ff'' = 0$ Rewrite as
  
  \[
  f' = G \\
  G' = H \\
  H' = -\frac{1}{2}f H
  \]

- Want an initial value problem. $f(0) = G(0) = 0$
- $H(0)$ not known $\rightarrow$ Use SHOOTING technique
- Approximate $\eta = \infty$ @ $\eta = 10$
# Velocity profile - Blasius flow

![Solution of the Blasius equation](image)

**Figure:** Velocity profile for Blasius flow
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Thermal Boundary layer equation

- Similarity solution → general technique. Other problems as well.
- Velocity Boundary layer - Convective Vs Viscous momentum transport
- Thermal Boundary layer - Convective Vs Conductive heat transfer
The 2D Steady state energy equation without pressure work and viscous dissipation

\[ u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = k \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right) \]

\[ \psi = \sqrt{U_\infty \nu x} \cdot f(\eta) \]

\[ \theta = \frac{T - T_w}{T_\infty - T_w} \quad \eta = \frac{y}{\sqrt{\frac{\nu x}{U_\infty}}} \]

Order of mag analysis → \( \frac{\partial^2 T}{\partial x^2} \) can be neglected
Transformation

- $T_w = \text{constant}$ - Simplifying assumption

\[
\frac{\partial \theta}{\partial x} = \frac{\partial \theta}{\partial \eta} \frac{\partial \eta}{\partial x} = \theta' \left( -\frac{1}{2x} \sqrt{\frac{U_e}{\nu x}} y + \frac{1}{2} \frac{u}{\sqrt{U_e \nu x}} \frac{dU_e}{dx} \right)
\]

\[
\frac{\partial \theta}{\partial y} = \frac{\partial \theta}{\partial \eta} \frac{\partial \eta}{\partial y} = \theta' \left( \sqrt{\frac{U_e}{\nu x}} \right) \quad \frac{\partial^2 \theta}{\partial y^2} = \frac{U_e}{\nu x} \theta''
\]
Transformation

- $T_w = constant$ - Simplifying assumption

$$\frac{u_e \theta''}{x \Pr} + \frac{f \theta'}{2} \left( \frac{u_e}{x} + \frac{du_e}{dx} \right) = 0$$

- Choose $\frac{du_e}{dx} = m \frac{u_e}{x}$

$$\theta'' + \frac{Pr}{2} (m + 1) f \theta' = 0$$
Solution

- $f \rightarrow$ Known from previous problem
- Let $\theta' = G$ Becomes first order ODE

$$G' + \frac{(m+1)Pr}{2} f G = 0$$

- Choose

$$I.F = \exp \left( \int_0^\eta \frac{m+1}{2} Pr f d\eta \right)$$

- We get

$$\theta' = C_1 \exp \left( - \int_0^\eta \frac{m+1}{2} Pr f d\eta \right)$$

- Integrate again and apply B.C’s
Final solution

\[ \theta = \frac{\int_0^n \exp \left( -\int_0^n \frac{m+1}{2} Pr f \, d\eta \right) \, d\eta}{\int_0^\infty \exp \left( -\int_0^n \frac{m+1}{2} Pr f \, d\eta \right) \, d\eta} \]
Summary

- Self-similar solutions to PDE’s can be obtained.
- Appropriate scaling of co-ordinates.
- Convert PDE’s to ODE’s
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