Classification of partial differential equations and their solution characteristics

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Introduction to Differential Equations

• Today: An example involving your bank account, and nice pictures called slope fields. How to read a differential equation.

• A differential equation is a relation between the independent, dependent variables and their differential coefficients.

• Example: \( x^2 \frac{d^2y}{dx^2} + xy \frac{dy}{dx} + e^y = 0 \)

• The order of differential equation is defined to be the order of the highest order derivative of the dependent variable occurring in the differential equation.

• Example: \( \frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} + \frac{dy}{dx} = 0 \) is a differential equation of order 3

• The degree of a differential equation is the highest power of the highest order derivative after making the equation free from radicals and fractional indices as far as the derivatives are concerned

• Example: \( (i) \left( \frac{d^2y}{dx^2} \right)^3 = 2x^2 + 7\sqrt{x} \) Order-2 , Degree- 3.
In mathematics, an **ordinary differential equation** (or **ODE**) is a relation that contains functions of only one independent variable, and one or more of their derivatives with respect to that variable.

i.e. differential coefficients have reference to a single independent variable

A simple example is Newton's second law of motion, which leads to the differential equation

\[
m \frac{d^2 x(t)}{dt^2} = F(x(t)),
\]
The partial differential equations (PDE) are a type of differential equation, i.e., a relation involving an unknown function (or functions) of several independent variables and their partial derivatives with respect to those variables.

- i.e. there are two or more independent variables and partial differential coefficients with respect to any of them.
- Problems involving functions of several variables; such as the propagation of sound or heat, electrostatics, fluid flow, and elasticity.
Representation

- A partial differential equation (PDE) for the function $u(x_1, \ldots x_n)$ is of the form

$$F(x_1, \cdots x_n, u, \frac{\partial}{\partial x_1}u, \cdots \frac{\partial}{\partial x_n}u, \frac{\partial^2}{\partial x_1 \partial x_1}u, \frac{\partial^2}{\partial x_1 \partial x_2}u, \cdots) = 0$$

$$F(x, y, \cdots, u, u_x, u_y, \cdots, u_{xx}, u_{xy}, \cdots) = 0$$
Partial Differential Equations

- A relatively simple partial differential equation is
  \[ u_x(x, y) = 0 \]
- General solution of the above equation is
  \[ u(x, y) = f(y) \]
- General solution involves arbitrary functions

Ordinary Differential Equations

- The analogous ordinary differential equation is
  \[ u'(x) = 0 \]
- General solution of the above equation is
  \[ u(x) = c \]
- General solution involves arbitrary constants
Differential equations can be classified as

- Linear
  - Homogenous
  - Non homogeneous
- Non linear
• **Linear PDE**
  • A PDE in $u$ is linear if the coefficients of the term containing $u$ and its derivative are independent of $u$. These can at most depend upon the independent variables $x$ and $y$.
  • i.e. if it can be expressed in the following form
    \[ a(x, y) \frac{\partial u(x, y)}{\partial x} + b(x, y) \frac{\partial u(x, y)}{\partial y} + c(x, y) u(x, y) = d(x, y). \]
• **Homogeneous PDE** —each term of the linear PDE contains either the dependent variable or its derivative
• Non homogenous
• Non linear

• A PDE in $u$ is non linear if the coefficients of the term containing $u$ and its derivative are dependent on $u$

$$a(x, y, u(x, y)) \frac{\partial u(x, y)}{\partial x} + b(x, y, u(x, y)) \frac{\partial u(x, y)}{\partial y} = c(x, y, u(x, y)).$$
Second order linear PDE

- Parabolic
- Elliptical
- Hyperbolic
Boundary and Initial conditions

- There are a number of solutions which exist for a given PDE.
- Therefore, PDE's are usually specified through a set of boundary or initial conditions.
- A **boundary condition** expresses the behavior of a function on the boundary (border) of its area of definition.
- An **initial condition** is like a boundary condition, but only for one direction.
What are boundary conditions?

1. The mathematician's point of view:
   domain
   ➔ differential equation
   ➔ boundary conditions

2. The physicist's point of view:
   differential equation ➔ physics
   Domain ➔ space
   boundary conditions ➔ influence of outer world

3. The software engineer's point of view:
   differential equation ➔ expression of differentials
   domain ➔ grid
Boundary Conditions

Explicit Boundary Conditions

**Dirichlet boundary condition**
value of $u$ on the boundary

**von Neumann boundary conditions**
Specifying the component normal to the Gradient

**Mixed (Robin’s) boundary conditions**
Mixture of above two

Implicit Boundary Conditions

• we have certain conditions we wish to be satisfied

• This is usually the case for systems defined on an infinite definition area.

• Eg- in Schrödinger equation, we require the wave function to be normalisable. The wave function is thus allowed to blow up at infinity

• Eg- assumption of continuity and differentiability
Domain of influence and domain of dependence

- Consider the solution to the parabolic heat equation
- At time $t_p$, the B.C.’s at $x=0$ and $L$ influences the solution at $x_p$. The solution at $(t_p, x_p)$ also influences the solution in the future. Thus semi-infinite strip above $t=t_p$ is called domain of influence and all the points where $t \leq t_p$ comprises of domain of dependence
Second order linear PDE

- Parabolic
- Elliptical
- Hyperbolic
Linear Second Order PDE in two independent variable

Second order PDE can be classified according to the coefficients of highest order term

\[ A \frac{\partial^2 u}{\partial x^2} + B \frac{\partial^2 u}{\partial x \partial y} + C \frac{\partial^2 u}{\partial y^2} + D \frac{\partial u}{\partial x} + E \frac{\partial u}{\partial y} + Fu + G = 0 \]

The PDE is

- Elliptic if \( B^2 - 4AC < 0 \)
- Parabolic if \( B^2 - 4AC = 0 \)
- Hyperbolic if \( B^2 - 4AC > 0 \)
If there are \( n \) independent variables \( x_1, x_2, \ldots, x_n \), a general linear partial differential equation of second order has the form

\[
Lu = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{i,j} \frac{\partial^2 u}{\partial x_i \partial x_j} + \text{plus lower order terms} = 0.
\]

The classification depends upon the signature of the eigenvalues of the coefficient matrix.

- **Elliptic**: The eigenvalues are all positive or all negative.
- **Parabolic**: The eigenvalues are all positive or all negative, save one that is zero.
- **Hyperbolic**: There is only one negative eigenvalue and all the rest are positive, or there is only one positive eigenvalue and all the rest are negative.
- **Ultrahyperbolic**: There is more than one positive eigenvalue and more than one negative eigenvalue, and there are no zero eigenvalues. There is only limited theory for ultra hyperbolic equations.
Example

\[ x^2 \frac{\partial^2 u}{\partial x^2} + y^2 \frac{\partial^2 u}{\partial y^2} + z^2 \frac{\partial^2 u}{\partial z^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + 2xz \frac{\partial^2 u}{\partial x \partial z} + 2yz \frac{\partial^2 u}{\partial y \partial z} = 0 \]

The matrix associated with this equation is

\[
\begin{pmatrix}
x^2 & xy & xz \\
xy & y^2 & yz \\
xz & yz & z^2 \\
\end{pmatrix}
\]

If we evaluate its characteristic polynomial we find that it is

\[ \lambda(x^2 - y^2 + z^2 - \lambda) = 0 \]

Since this has always (for all \(x, y, z\)) two zero eigenvalues this is a parabolic differential equation.
Solution of Partial Differential Equation

Find a Function which

- is continuous on the boundary of D
- has those derivatives in the interior of D
- satisfies the equation in the interior of D

*Where D is some region defined by independent variables*
<table>
<thead>
<tr>
<th>Type</th>
<th>Sign of ( b^2 - 4ac )</th>
<th>Example</th>
<th>Solution region</th>
</tr>
</thead>
<tbody>
<tr>
<td>Elliptic</td>
<td>-</td>
<td>Laplace’s equation: ( \Phi_{xx} + \Phi_{yy} = 0 )</td>
<td>Closed</td>
</tr>
<tr>
<td>Hyperbolic</td>
<td>+</td>
<td>Wave equation: ( u^2 \Phi_{xx} = \Phi_{tt} )</td>
<td>Open</td>
</tr>
<tr>
<td>Parabolic</td>
<td>0</td>
<td>Diffusion equation: ( \Phi_{xx} = k \Phi_t )</td>
<td>Open</td>
</tr>
</tbody>
</table>
Classification by characteristics

- Consider the following general second-order quasi-linear PDE
  \[ A(u_x, u_y, u, x, y)u_{xx} + B(u_x, u_y, u, x, y)u_{xy} + C(u_x, u_y, u, x, y)x_{yy} = \Phi(u_x, u_y, u, x, y) \]
- we will need to find two characteristics because the equation is of second-order
- Assuming \( u, u_x, \) and \( u_y \) are specified as initial conditions
- Consider such a characteristic line in the \((x, y)\) plane specified by the differential increment \((dx, dy)\) along the line.
• Question — unique determination of $u_{xx}$, $u_{yy}$ from PDE and initial conditions

• By differentiating $u_x$ and $u_y$ along the assumed characteristic line, we get

$$d\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial x^2} \, dx + \frac{\partial^2 u}{\partial x \partial y} \, dy$$

$$d\left(\frac{\partial u}{\partial y}\right) = \frac{\partial^2 u}{\partial x \partial y} \, dx + \frac{\partial^2 u}{\partial y^2} \, dy$$

$$\Phi(u_x, u_y, u, x, y) = A u_{xx} + B u_{xy} + C u_{yy}$$

$$\begin{pmatrix} dx & dy & 0 \\ 0 & dx & dy \\ A & B & C \end{pmatrix} \begin{pmatrix} \frac{\partial^2 u}{\partial x^2} \\ \frac{\partial^2 u}{\partial x \partial y} \\ \frac{\partial^2 u}{\partial y^2} \end{pmatrix} = \begin{pmatrix} d\left(\frac{\partial u}{\partial x}\right) \\ d\left(\frac{\partial u}{\partial y}\right) \\ \Phi(u_x, u_y, u, x, y) \end{pmatrix}$$
The solution will be non-determinant if

\[
\begin{vmatrix}
  dx & dy & 0 \\
  0 & dx & dy \\
  A & B & C \\
\end{vmatrix} = C(dx)^2 + B(dx)(dy) + A(dy)^2 = 0
\]

The solution is given as

\[
\left(\frac{dy}{dx}\right)^\pm = \frac{B \pm \sqrt{B^2 - 4AC}}{2A}
\]

Which is real only for \(B^2 - 4AC > 0\). This is called characteristic equation.

Hyperbolic - two distinct characteristics

Parabolic – two characteristics degenerate into one which is incomplete for second order PDE – initial value problem

Elliptic- No real characteristics – boundary problem,
General relation between the physical problems and different type of PDE’s

- Propagation problems lead to parabolic or hyperbolic PDE’s.
- Equilibrium equations lead to elliptic PDE.
- Most fluid equations with an explicit time dependence are Hyperbolic PDEs
- For dissipation problem, Parabolic PDEs
Elliptical Equations

- Characteristics- no real characteristics
- $B^2 - 4AC < 0$
- Eigen Values- all positive or all negative.
- Initial Conditions- no initial conditions
- Boundary Conditions- any type
Elliptical

- Elliptic equations produce stationary and energy-minimizing solutions,
- The simplest example of an elliptic equation is the Laplace equation
  \[
  \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = 0,
  \]
  where \( x \) and \( y \) play the role of the spatial coordinates. Note that the highest derivative terms in equation have like signs. The Laplace equation is often written briefly as \( \Delta(w) = 0 \), where \( \Delta \) is the Laplace operator.
- The Laplace equation in heat and mass transfer theory, fluid mechanics, elasticity, electrostatics, and other areas of mechanics and physics. For example, in heat and mass transfer theory, this equation describes steady-state temperature distribution in the absence of heat sources and sinks in the domain under study.
- A solution to the Laplace equation is called a harmonic function.
• Other examples of elliptic equations include the nonhomogeneous Poisson's equation

\[ \nabla^2 u = f(x) \]

• boundary conditions are used to give the constraint \( u(x, y) = g(x, y) \) on, where 

\[ u_{xx} + u_{yy} = f\left(u_x, u_y, u, x, y\right) \]

• property of constant coefficient elliptic equations ➔

• their solutions can be studied using the Fourier transform
• the solutions of elliptic PDEs are always smooth
• even if the initial and boundary conditions are rough (sharp corners)
• boundary data at any point affect the solution at all points in the domain

• Region of influence – entire domain
• Region of dependence – entire domain

Entire Region is the Domain of Influence and Dependence
• Laplace Equation- \( \nabla^2 \varphi = 0 \)
• Fundamental solution of Laplace’s Equation-
  \[ \Delta u = u_{xx} + u_{yy} + u_{zz} = -\delta(x - x', y - y', z - z'), \]
• In Cartesian coordinates- \( \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0. \)
Numerical Solutions to Laplace’s Equation

- **Finite – Difference Method**
  - Finite-difference methods superimpose a regular grid on the region of interest.
  - Approximate Laplace’s equation at each grid-point.
  - The resulting equations are solved by iteration.

- **Finite Elements Method**
  - Divide the problem of interest into a mesh of geometric shapes called finite elements.
  - The potential within an element is described by a function that depends on its values at the cell corners and parameters defining the state of the element.
  - Several such cells are assembled to solve the entire problem.
  - The solution can be refined by subdividing the regions of the mesh.
Physical Problem

- Consider a thin metal square plate with dimensions 0.5 meters by 0.5 meters. Two adjacent boundaries are held at a constant 0 deg C. The heat on the other two boundaries increases linearly from 0 deg C to 100 deg C. We want to know what the temperature is at each point when the temperature of the metal has reached steady-state.
Parabolic Equations

- Characteristics: only one characteristic
- \( B^2 - 4AC = 0 \)
- Eigen Values: all positive or all negative, save one that is zero.
- Initial Conditions: Dirichlet condition
- Boundary Conditions: mixed conditions
Parabolic Equations

- parabolic equations produce a smooth-spreading flow of an initial disturbance

- **Example 1** - The simplest example of a parabolic equation is the heat equation

\[
\frac{\partial w}{\partial t} - \frac{\partial^2 w}{\partial x^2} = 0,
\]

- where the variables \( t \) and \( x \) play the role of time and a spatial coordinate, respectively. Note that equation contains only one highest derivative term
Example 2- A (one-dimensional) parabolic PDE of the form and is used in the study of gas diffusion and often referred to as the diffusion equation.

Initial-boundary conditions are used to give

\[ u(x, t) = g(x, t) \text{ for } x \in \partial \Omega, \ t > 0 \]

\[ u(x, 0) = v(x) \text{ for } x \in \Omega, \]
- Region of influence: Part of domain away from initial data line from the characteristic curve
- Region of dependence: Part of domain from the initial data line to the characteristic curve
• usually time dependent and represent diffusion-like processes.
• Solutions are smooth in space but may possess singularities
• However, information travels at infinite speed in a parabolic system.
Solution of the heat equation: separation of variables

- The general Equation \( \frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \),
- Boundary conditions- \( u(0,t) = 0, \quad u(l,t) = 0, \)
- Initial conditions- \( u(x,0) = f(x), \)
- The temperature is assumed to be independent of \( x \) and \( t \)
  \( u(x,t) = X(x)T(t) \)
- On substitution we get – \( \dot{T} = c^2 kT, \)
- Where \( k \) is separation constant. So we have \( \frac{X''}{X} = c^2 \frac{\dot{T}}{T} = k, \)
- With solution
  \( T(t) = T_0 e^{c^2 kt}. \)
t is obviously unphysical for the temperature to increase in time without any additional heating mechanism and so we must assume that $k$ is negative.

$$k = -p^2.$$ 

$$X'' + p^2 X = 0,$$

which is the simple harmonic motion equation with trigonometric solutions.

Thus the solution

$$X(x) = A \cos px + B \sin px.$$ 

Applying initial and boundary conditions, we get

$$u(x, t) = \sum_{n=0}^{\infty} B_n \sin \frac{n\pi}{l} xe^{-n^2\pi^2 c^2 t/l^2}.$$
• Applying Fourier series,

\[ B_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi}{l} x \, dx, \]
Physical Problem

- Consider a rod of length \( l \) that is perfectly insulated. We want to examine the flow of heat along the rod over time. The two ends of the rod are held at 0 deg C. We denote heat as temperature, and seek the solution to the equation \( u(x, t) \): the temperature of \( x \) at time \( t > 0 \).
Hyperbolic Equations

- Characteristics - two characteristics
- $B^2 - 4AC > 0$
- Eigen Values - only one negative eigenvalue and all the rest are positive, or there is only one positive eigenvalue and all the rest are negative.
Hyperbolic Equations

- hyperbolic equations produce a propagating disturbance
- The simplest example of a hyperbolic equation is the wave equation
  \[
  \frac{\partial^2 w}{\partial t^2} - \frac{\partial^2 w}{\partial x^2} = 0,
  \]
- where the variables \( t \) and \( x \) play the role of time and the spatial coordinate
- This equation is also known as the equation of vibration of a string. It is often encountered in elasticity, aerodynamics, acoustics, and electrodynamics
- The general solution of the wave equation is \( u(x, t) = F(x - ct) + G(x + ct) \).
- This solution has the physical interpretation of two traveling waves of arbitrary shape that propagate to the right and to the left along the \( -x \)-axis with a constant speed equal to \( c \).
• Initial-boundary conditions are used to give
  
  \[ u(x, y, t) = g(x, y, t) \text{ for } x \in \partial \Omega, \ t > 0 \]
  
  \[ u(x, y, 0) = v_0(x, y) \text{ in } \Omega \]
  
  \[ u_t(x, y, 0) = v_1(x, y) \text{ in } \Omega, \]
  
• where \[ u_{xy} = f(u_x, u_t, x, y) \]
• the smoothness of the solution depends on the smoothness of the initial and boundary conditions
• information travels at a finite speed called the wavespeed
• Region of influence: Part of domain, between the characteristic curves, from point $P$ to away from the initial data line
• Region of dependence: Part of domain, between the characteristic curves, from the initial data line to the point $P$
D'Alembert's solution of the Wave Equation

- change coordinates from $x$ and $t$ to $\xi$ and $\eta$
  \[ \xi = x - ct \quad \text{and} \quad \eta = x + ct. \]
- Thus
  \[ u(x,t) = u(\xi, \eta) = u(\xi(x,t), \eta(x,t)) \]
- Hence,
  \[ \frac{\partial u}{\partial x} = \frac{\partial \xi}{\partial x} \frac{\partial u}{\partial \xi} + \frac{\partial \eta}{\partial x} \frac{\partial u}{\partial \eta} = \frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta} \]
  \[ \frac{\partial u}{\partial t} = \frac{\partial \xi}{\partial t} \frac{\partial u}{\partial \xi} + \frac{\partial \eta}{\partial t} \frac{\partial u}{\partial \eta} = -c \frac{\partial u}{\partial \xi} + c \frac{\partial u}{\partial \eta}. \]
- Taking second derivative
  \[ \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial \xi^2} + 2 \frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{\partial^2 u}{\partial \eta^2} \]
  \[ \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial \xi^2} - 2c^2 \frac{\partial^2 u}{\partial \xi \partial \eta} + c^2 \frac{\partial^2 u}{\partial \eta^2}. \]
• Solving the above equations, we get
  \[ \frac{\partial^2 u}{\partial \xi \partial \eta} = 0. \]
  \[
  \Rightarrow \frac{\partial^2 u}{\partial \xi \partial \eta} = 0.
  
• Integrating w.r.t. $\xi$
  \[ \frac{\partial u}{\partial \eta} = g(\eta) \]

• where $g(\eta)$ is an arbitrary function of $\eta$

• Integrating w.r.t $\eta$, 
  \[ u(\xi, \eta) = F(\xi) + G(\eta) \]

• where $F(\xi)$ is an arbitrary function of $\xi$

• And $G(\eta) = \int g(\eta) d\eta$

• Finally in terms of $x$ and $y$, 
  \[ u(x, t) = F(x - ct) + G(x + ct). \]
Physical Problem

- An elastic string is stretched between two horizontal supports that are separated by length \( l \). The PDE defines vertical displacement of the string at time \( t \). (We are assuming that when we “pluck” the string, it only vibrates in one direction.)
Numerical methods to solve PDEs

- The three most widely used numerical methods to solve PDEs are the finite element method (FEM), finite volume methods (FVM) and finite difference methods (FDM).
Summary

- Differential Equations
- Classification (ordinary and partial)
- Boundary and initial conditions
- Classification (parabolic, elliptical, hyperbolic)
- Solution of Heat Equation
- Solution of Wave equation
References

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- Wikipedia
THANK YOU