SHRUTI JAIN
B. Tech III Year, Electronics and Communication
IIT Roorkee

Tutors:
Professor G. Biswas
Professor S. Chakraborty
ACKNOWLEDGMENTS

I would like to thank my mentor, Professor G. Biswas without whose help, this presentation would not have been possible. My presentation is a result of his lecture notes and his consistent guidance.

Participating in the winter academy has been a learning experience and I would like to thank all the members of the academy for giving me this opportunity. It has been a pleasure working on this presentation.
FINITE VOLUME METHOD: AN OUTLINE

• Introduction to Finite Volume Method
• 2 approaches
  - Weighted Residual Approach
  - Physical Approach
• Boundary Conditions in Finite Volume Method
• Equations with First Order Derivatives
• Equations with Second Order Derivatives
Finite Volume Method is a sub domain method with piecewise definition of the field variable in the neighborhood of chosen control volumes.

The total solution domain is divided into many small control volumes which are usually rectangular in shape.

Nodal points are used within these control volumes for interpolating the field variable and usually, a single node at the centre of the control volume is used for each control volume.

This method was developed by Patankar and Spalding and they proposed the use of physical approach for deriving the nodal equations.
Consider a 2-D, steady heat conduction in rectangular geometry (Figure 1). The 2-D heat conduction equation is

\[ k \nabla^2 T + Q = 0 \]  \hspace{1cm} (1)

Where \( T(x,y) \) is the temperature field, \( k \) is the thermal conductivity and \( Q \) is the heat generation per unit volume.

The two alternative ways of setting up the nodal equations are the weighted residual approach and the physical approach.
WEIGHTED RESIDUAL APPROACH

Weighted residual approach for 2-D heat conduction equation:

\[
\int \int_{\Omega} W_i (k \nabla^2 T + Q) \, dx \, dy = 0
\]

(2)

Where the weight

\( W_i = 1 \) within the \( i^{th} \) control volume \( \Omega_i \)

\( W_i = 0 \) outside the \( i^{th} \) control volume \( \Omega_i \)

Thus, we get, for each \( i = 1, \ldots, n \)

\[
\int \int_{\Omega_i} (k \nabla^2 T + Q) \, dx \, dy = 0
\]

(3)

Integrating the above equation by parts, we get:

\[
\int \int \int_{\Omega_i} (\nabla \cdot k \nabla T + Q) \, dx \, dy = \oint_{c_i} k \frac{\partial T}{\partial n} \, dl + \int \int_{\Omega_i} Q \, dx \, dy = 0
\]
Where the Gauss divergence theorem has been used to convert the volume integral to a surface integral.

\[
\int_{\Omega_i} \int Q \, dx \, dy = \oint_{c_i} -k \frac{\partial T}{\partial n} \, dl
\]  

(4)
WEIGHTED RESIDUAL APPROACH

The meaning of Equ. (4) is that the net heat generation rate \( = \int \int Q \, dx \, dy \) in the control volume is equal to the net sum of the rate of heat energy going out of the control volume \( \Omega_i \) \( = - \int \int k \, \frac{\partial T}{\partial n} \, dl \),

where \( c_i \) is the boundary of the control volume \( \Omega_i \).
PHYSICAL APPROACH

Figure 2: Balance of Heat Flux in a Control Volume
PHYSICAL APPROACH

Considering the balance of heat flux in Figure 2 for the control volume (for unit depth in z direction):

\[ q_e \cdot \Delta y \cdot 1 - q_w \cdot \Delta y \cdot 1 + q_n \cdot \Delta x \cdot 1 - q_s \cdot \Delta x \cdot 1 = Q \cdot 1 \cdot \Delta x \Delta y \]

Net rate of heat leaving the control volume through the boundary = Rate of heat generation within the control volume (CV) at steady state:

\[ \sum \vec{q} \cdot d\vec{s} = Q \Delta x \Delta y \]  \hspace{1cm} (5)

which is the same equation as obtained before.
PHYSICAL APPROACH

In the implementation of the finite volume method, the heat fluxes are represented in terms of nodal temperatures (e.g. $T_E$ etc. at the CV centers). Thus, assuming temperature to have linear variation between points E and P, the heat flux $q_e$ can be evaluated as follows:

$$q_e = -k \frac{\partial T}{\partial x} \bigg|_e = -\frac{k \{T_E - T_P\}}{\Delta x}$$

(6)

While deriving equation (6) it has been assumed that the cell size is constant in x-direction. Similarly, $q_w$ is given by

$$q_w = -k \frac{\partial T}{\partial x} \bigg|_w = -\frac{k \{T_P - T_W\}}{\Delta x}$$

(7)
PHYSICAL APPROACH

Using similar expressions for \( q_n \) and \( q_s \) also, the nodal equation for point P becomes:

\[
+ \frac{k}{\Delta x} \left\{ -T_E + T_P \right\} + \frac{k}{\Delta x} \left\{ T_P - T_W \right\} + \frac{k}{\Delta y} \left\{ T_P - T_N \right\} \\
+ \frac{k}{\Delta y} \left\{ T_P - T_S \right\} = Q \Delta x \Delta y
\]

This equation can be rewritten in the familiar form used in finite difference as:

\[
T_P \left\{ 2 + 2\beta^2 \right\} - T_E - T_W - \beta^2 T_N - \beta^2 T_S = \frac{Q \Delta x^2}{k}
\]

Where,

\[
\beta = \frac{\Delta x}{\Delta y}
\]

During numerical implementation, the subscripts E, W, etc. will be changed to numerical indices of i, j and solved in the same way.
BOUNDARY CONDITIONS: When heat flux at a boundary is specified

The control volumes adjacent to the \( x = 0 \) boundary as shown Figure 3, the term \( q_w \) will be substituted by \( q_0 \) in equation (5). Thus,

\[
q_e \Delta y - (+q_0) \Delta y + q_n \Delta x - q_s \Delta x
= -\frac{k \{T_E - T_P\}}{\Delta x} \Delta y - q_0 \Delta y - \frac{k \{T_N - T_P\}}{\Delta y} \Delta x
+ \frac{k \{T_P - T_S\}}{\Delta y} \Delta x = Q \Delta x \Delta y
\]

(10)
BOUNDARY CONDITIONS: When boundary temperature is specified

When boundary temperature is specified, the control volume shapes near the boundary can be changed to facilitate the implementation of the boundary conditions.

For example: consider the BC, \( T = T_L \) at boundary \( x = L \). For nodes on boundary an imaginary extension of CV’s outside the actual domain can be considered.

We consider the boundary to be the center of the boundary cells and their width to be \( \Delta x / 2 \), thus reducing the width of the adjacent cells to \( 3\Delta x / 4 \).

Figure 4: Boundary Condition, at \( x = L, T = T_L \)
BOUNDARY CONDITIONS

The nodal equation for the adjacent cell $P$ will be written considering a shortened control volume

\[
q_e \cdot \Delta y - q_w \cdot \Delta y + q_n \Delta x_1 - q_s \Delta x_1 = -k \frac{(T_L - T_P)}{(\Delta x/2)} \Delta y + k \frac{(T_P - T_W)}{\Delta x} \Delta y - k \frac{(T_N - T_P)}{\Delta y} \Delta x_1 + k \frac{(T_P - T_s)}{\Delta y} \Delta x_1 = Q \Delta x_1 \Delta y
\]

(11)

Where $\Delta x_1 = \frac{3}{4} \Delta x$.

Note that $T_L$ has been substituted instead of $T_W$ is the above equation, so the boundary condition is being directly applied.
EQUATION WITH FIRST DERIVATIVES

General first-order equation:

\[ \frac{\partial E}{\partial t} + \frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} = 0 \]  \hspace{1cm} (12)

For \( E = \rho u, \; F = \rho u^2, \; G = \rho uv \),

the above equation is the inviscid momentum equation in the x-directions.

In a similar manner, for x direction viscous momentum equation,

\[ E = \rho \ u, \quad F = \left( \rho u^2 + p - \mu \frac{\partial u}{\partial x} \right), \quad G = \left( \rho \ u \ v - \mu \frac{\partial u}{\partial y} \right) \]  \hspace{1cm} (13)
Figure-5: Finite Volume Mesh System
We consider the area integral of general first order equation over $\Omega_p$

$$\int \int_{ABCD} \left( \frac{\partial E}{\partial t} + \frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} \right) \, dx \, dy = 0 \quad (14)$$

$\Omega_p$ is the finite volume (quadrilateral) ABCD shown in figure 5

Green’s theorem:

$$\int \int_{ABCD} \left( \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) \, dx \, dy = \oint_c f \, dx + g \, dy \quad (15)$$
Using Green’s theorem the equation (14) becomes:

\[
\frac{d}{dt} \int \int E \, dV + \int_{ABCD} \mathbf{H} \cdot \mathbf{n} \, ds = 0
\]  

(16)

where \( \mathbf{H} = (F, G) \) and \( \mathbf{n} \) is the outward unit normal of segment \( ds \). On a counter clockwise contour \( \mathbf{n} = \frac{dy \mathbf{i} - dx \mathbf{j}}{ds} \), where \( ds = \sqrt{dx^2 + dy^2} \).

Hence \( \mathbf{H} \cdot \mathbf{n} \, ds = F \, dy - G \, dx \).

The finite volume method is a discretization of the governing equation in integral form, in contrast to the finite difference method, which is unusually applied to the governing equation in differential form.

\[
\frac{d}{dt} (A \, E_{i, j}) + (F \, \Delta y - G \, \Delta x)_{AB} + (F \, \Delta y - G \, \Delta x)_{BC} \\
+ (F \, \Delta y - G \, \Delta x)_{CD} + (F \, \Delta y - G \, \Delta x)_{DA} = 0
\]  

(17)
Where $A$ is the area of the quadrilateral $ABCD$ in Figure-5, and the average value of $E$ over the quadrilateral is represented by $E_{i,j}$ and the remaining terms are approximations for the line integral over segments $AB$, $BC$, $CD$ and $DA$ respectively.

$$\Delta y_{AB} = y_B - y_A, \quad \Delta x_{AB} = x_B - x_A \quad \text{and} \quad F_{AB} = 0.5 \left( F_{i, j-1} + F_{i, j} \right), \quad G_{AB} = 0.5 \left( G_{i, j-1} + G_{i, j} \right)$$
With similar expressions for $\Delta y_{BC}$, etc. If $A$ is not a function of time, then:

$$
\begin{align*}
A \frac{dE_{i,j}}{dt} &+ 0.5 \left( F_{i, j-1} + F_{i, j} \right) \Delta y_{AB} - 0.5 \left( G_{i, j-1} + G_{i, j} \right) \Delta x_{AB} \\
&+ 0.5 \left( F_{i, j} + F_{i+1, j} \right) \Delta y_{BC} - 0.5 \left( G_{i, j} + G_{i+1, j} \right) \Delta x_{BC} \\
&+ 0.5 \left( F_{i, j} + F_{i, j+1} \right) \Delta y_{CD} - 0.5 \left( G_{i, j} + G_{i, j+1} \right) \Delta x_{CD} \\
&+ 0.5 \left( F_{i-1, j} + F_{i, j} \right) \Delta y_{DA} - 0.5 \left( G_{i-1, j} + G_{i, j} \right) \Delta x_{DA} = 0
\end{align*}
$$

(18)

If the grid-mesh is uniform and coincides with lines of constant $x$ and $y$, the above equation (18) becomes:

$$
\frac{d}{dt} E_{i, j} + \frac{F_{i+1, j} - F_{i-1, j}}{2 \Delta x} + \frac{G_{i, j+1} - G_{i, j-1}}{2 \Delta y} = 0
$$

(19)

which coincides with a central difference representation for the spatial terms in equation (12)
Let us consider Laplace’s equation:

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$  \hspace{1cm} (20)

The finite volume method follows the application of the subdomain method to equation (20).

$$\int_{ABCD} \left( \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} \right) \, dx \, dy = \int_{ABCD} H \cdot n \, ds = 0$$

where Green’s theorem has again been used and now:

$$H \cdot n \, ds = \frac{\partial \phi}{\partial x} \, dy - \frac{\partial \phi}{\partial y} \, dx$$  \hspace{1cm} (21)
The above equation can be evaluated approximately over the segments AB, BC, CD and DA by:

\[
\begin{align*}
\left[ \frac{\partial \phi}{\partial x} \right]_{i, \; j-1/2} & \Delta y_{AB} - \left[ \frac{\partial \phi}{\partial y} \right]_{i, \; j-1/2} \Delta x_{AB} \\
+ \left[ \frac{\partial \phi}{\partial x} \right]_{i+1/2, \; j} & \Delta y_{BC} - \left[ \frac{\partial \phi}{\partial y} \right]_{i+1/2, \; j} \Delta x_{BC} \\
+ \left[ \frac{\partial \phi}{\partial x} \right]_{i, \; j+1/2} & \Delta y_{CD} - \left[ \frac{\partial \phi}{\partial y} \right]_{i, \; j+1/2} \Delta x_{CD} \\
+ \left[ \frac{\partial \phi}{\partial x} \right]_{i-1/2, \; j} & \Delta y_{DA} - \left[ \frac{\partial \phi}{\partial y} \right]_{i-1/2, \; j} \Delta x_{DA} = 0
\end{align*}
\]
EQUATIONS WITH SECOND DERIVATIVES

Figure 4: Finite Volume for a Curvilinear Grid

Here \([\partial \phi / \partial x]_{i, j-1/2}\) is evaluated as the mean value over the area \(B' B C' D' A A' B'\) in Figure 4.
Thus we can write:

\[
\left[ \frac{\partial \phi}{\partial x} \right]_{i,j-1/2} = \left( \frac{1}{S_{A'B'C'D'}} \right) \int \int \left( \frac{\partial \phi}{\partial x} \right) \, dx \, dy = \left( \frac{1}{S_{A'B'C'D'}} \right) \int \phi \, dy
\]  

(23)

\[
\left[ \frac{\partial \phi}{\partial y} \right]_{i,j-1/2} = \left( \frac{1}{S_{A'B'C'D'}} \right) \int \int \left( \frac{\partial \phi}{\partial y} \right) \, dx \, dy = - \left( \frac{1}{S_{A'B'C'D'}} \right) \int \phi \, dx
\]  

(24)

Using Green’s theorem:

\[
\int_{A'B'C'D'} \phi \, dy \approx \phi_{i,j-1} \Delta y_{A'B'} + \phi_B \Delta y_{B'C'} + \phi_{i,j} \Delta y_{C'D'} + \phi_A \Delta y_{D'A'}
\]

A similar expression can be written for

\[
\int_{A'B'C'D'} \phi \, dx = \phi_{i,j-1} \Delta x_{A'B'} + \phi_B \Delta x_{B'C'} + \phi_{i,j} \Delta x_{C'D'} + \phi_A \Delta x_{D'A'}
\]
If the mesh is not too distorted,

\[ \Delta y_{A'B'} \approx -\Delta y_{C'D'} \approx \Delta y_{AB} \]
\[ \Delta y_{B'C'} \approx -\Delta y_{D'A'} \approx \Delta y_{j-1,j} \]
\[ \Delta x_{A'B'} \approx -\Delta x_{C'D'} \approx \Delta x_{AB} \]
\[ \Delta x_{B'C'} \approx -\Delta x_{D'A'} \approx \Delta x_{j-1,j} \]

\[ S_{A,B} = S_{A'B'C'D'} = |A'B'X'B'C'| \]

Hence,

\[ S_{AB} = \Delta x_{AB} \Delta y_{j-1,j} - \Delta y_{AB} \Delta x_{j-1,j} \]  \hspace{1cm} (25)
EQUATIONS WITH SECOND DERIVATIVES

Therefore the equation (23) & (24) becomes:

\[
\left[ \frac{\partial \phi}{\partial x} \right]_{i, j-1/2} = \frac{\Delta y_{AB} (\phi_{i, j-1} - \phi_{i, j}) + \Delta y_{j-1, j} (\phi_B - \phi_A)}{S_{AB}} \tag{26}
\]

\[
\left[ \frac{\partial \phi}{\partial y} \right]_{i, j-1/2} = -\left[ \Delta x_{AB} (\phi_{i, j-1} - \phi_{i, j}) + \Delta x_{j-1, j} (\phi_B - \phi_A) \right] \frac{1}{S_{AB}} \tag{27}
\]

Now, the first line of equation (22) can be evaluated as:

\[
\frac{(\Delta x_{AB}^2 + \Delta y_{AB}^2)(\phi_{i, j-1} - \phi_{i, j})}{S_{AB}} + \frac{(\Delta x_{AB} \Delta x_{j-1, j} + \Delta y_{AB} \Delta y_{j-1, j})(\phi_B - \phi_A)}{S_{AB}} \tag{28}
\]
In a similar manner, one can evaluate the following:

\[ \frac{\partial \phi}{\partial x} \bigg|_{i+1/2, j} = \frac{1}{S_{A'' B'' C'' D''}} \int \int \frac{\partial \phi}{\partial x} \, dx \, dy = \frac{1}{S_{A'' B'' C'' D''}} \oint \phi \, dy \]

\[ \frac{\partial \phi}{\partial x} \bigg|_{i+1/2, j} = \frac{1}{S_{A'' B'' C'' D''}} \int \int \frac{\partial \phi}{\partial y} \, dx \, dy = -\frac{1}{S_{A'' B'' C'' D''}} \oint \phi \, dx \]

\[ \oint_{A'' B'' C'' D''} \phi \, dy \approx \phi_B \Delta y_{A'' B''} + \phi_{i+1, j} \Delta y_{B'' C''} + \phi_C \Delta y_{C'' D''} + \phi_{i, j} \Delta y_{D'' A''} \]

and

\[ \oint_{A'' B'' C'' D''} \phi \, dx \approx \phi_B \Delta x_{A'' B''} + \phi_{i+1, j} \Delta x_{B'' C''} + \phi_C \Delta x_{C'' D''} + \phi_{i, j} \Delta x_{D'' A''} \]
EQUATIONS WITH SECOND DERIVATIVES

If the mesh is not too distorted,

\[ \Delta y_{A''B''} = -\Delta y_{C''D''} = \Delta y_{i, i+1} \quad \text{and} \quad \Delta y_{B''C''} = -\Delta y_{D''A''} = \Delta y_{BC} \]

\[ \Delta x_{A''B''} = -\Delta x_{C''D''} = \Delta x_{i, i+1} \quad \text{and} \quad \Delta x_{B''C''} = -\Delta x_{D''A''} = \Delta x_{BC} \]

The equivalent expressions for \([\partial \phi / \partial x]_{i, j+1/2}, [\partial \phi / \partial y]_{i, j+1/2}\) and other similar terms in equation (22), finally yields:

\[ M_{AB} \left( \phi_{i, j-1} - \phi_{i, j} \right) + N_{AB} \left( \phi_{B} - \phi_{A} \right) + M_{BC} \left( \phi_{i+1, j} - \phi_{i, j} \right) + N_{BC} \left( \phi_{C} - \phi_{B} \right) \]

\[ + M_{CD} \left( \phi_{i, j+1} - \phi_{i, j} \right) + N_{CD} \left( \phi_{D} - \phi_{C} \right) + M_{DA} \left( \phi_{i-1, j} - \phi_{i, j} \right) + N_{DA} \left( \phi_{A} - \phi_{D} \right) = 0 \]
EQUATIONS WITH SECOND DERIVATIVES

Where the geometrical parameters are:

\[
M_{AB} = \frac{\Delta x_{AB}^2 + \Delta y_{AB}^2}{S_{AB}}, \quad N_{AB} = \frac{\Delta x_{AB} \Delta x_{j-1, j} + \Delta y_{AB} \Delta y_{j-1, j}}{S_{AB}}
\]

\[
M_{BC} = \frac{\Delta x_{BC}^2 + \Delta y_{BC}^2}{S_{BC}}, \quad N_{BC} = \frac{\Delta x_{BC} \Delta x_{i+1, i} + \Delta y_{BC} \Delta y_{i+1, i}}{S_{BC}}
\]

\[
M_{CD} = \frac{\Delta x_{CD}^2 + \Delta y_{CD}^2}{S_{CD}}, \quad N_{CD} = \frac{\Delta x_{CD} \Delta x_{j+1, j} + \Delta y_{CD} \Delta y_{j+1, j}}{S_{CD}}
\]

and

\[
M_{DA} = \frac{\Delta x_{DA}^2 + \Delta y_{DA}^2}{S_{DA}}, \quad N_{DA} = \frac{\Delta x_{DA} \Delta x_{i-1, i} + \Delta y_{DA} \Delta y_{i-1, i}}{S_{DA}}
\]

In equation (31) \( \phi_A, \phi_B, \phi_C, \phi_D \) are evaluated as the average of the four surrounding nodal values. Thus

\[
\phi_A = 0.25 \left( \phi_{i, j} + \phi_{i-1, j} + \phi_{i-1, j-1} + \phi_{i, j-1} \right)
\]
Substitution into equation (31) generates the following nine-points discretization of equation (20):

\[
0.25 \ (N_{CD} - N_{DA}) \ \phi_{i-1, \ j+1} + [M_{CD} + 0.25 \ (N_{BC} - N_{DA})] \ \phi_{i, \ j+1} \\
+ \ 0.25 \ (N_{BC} - N_{CD}) \ \phi_{i+1, \ j+1} + [M_{DA} + 0.25 \ (N_{CD} - N_{AB})] \ \phi_{i-1, \ j} \\
- \ (M_{AB} + M_{BC} + M_{CD} + M_{DA}) \ \phi_{i, \ j} + [M_{BC} + 0.25 \ (N_{AB} - N_{CD})] \ \phi_{i+1, \ j} \\
+ \ 0.25 \ (N_{DA} - N_{AB}) \ \phi_{i-1, \ j-1} + [M_{AB} + 0.25 \ (N_{DA} - N_{BC})] \ \phi_{i, \ j-1} \\
+ \ 0.25 \ (N_{AB} - N_{BC}) \ \phi_{i+1, \ j-1} = 0
\]  

(32)

We need to calculate the geometric quantities like $M_{AB}$, $N_{AB}$ etc. only once for a given grid and can use the obtained values for all subsequent calculations.
Equation (32) is solved conveniently using a Successive Over-Relaxation (SOR) technique.

\[
\phi_{i, j}^* = \{0.25 \ (N_{CD} - N_{DA}) \ \phi_{i-1, j+1} + [M_{CD} + 0.25 \ (N_{BC} - N_{DA})] \ \phi_{i, j+1} + 0.25 \ (N_{BC} - N_{CD}) \ \phi_{i+1, j+1} + [M_{DA} + 0.25 \ (N_{CD} - N_{AB})] \ \phi_{i-1, j} + [M_{BC} + 0.25 \ (N_{AB} - N_{CD})] \ \phi_{i+1, j} + 0.25 \ (N_{DA} - N_{AB}) \ \phi_{i-1, j-1} + [M_{AB} + 0.25 \ (N_{DA} - N_{BC})] \ \phi_{i, j-1} + 0.25 \ (N_{AB} - N_{BC}) \ \phi_{i+1, j-1}\}^- \ / (M_{AB} + M_{BC} + M_{CD} + M_{D})
\]  

(33)

and the improved better value is:

\[
\phi_{i, j}^{n+1} = \phi_{i, j}^n + \lambda \ (\phi_{i, j}^* - \phi_{i, j}^n)
\]  

(34)

where \( \lambda \) is the relaxation parameter.

The discretised equation (32) reduces to the centre finite difference scheme on a uniform rectangular grid

\[
\frac{\phi_{i-1, j} - 2\phi_{i, j} + \phi_{i+1, j}}{\Delta x^2} + \frac{\phi_{i, j-1} - 2\phi_{i, j} + \phi_{i, j+1}}{\Delta y^2} = 0
\]  

(35)
SUMMARY

- Finite Volume Method is a sub domain method with piecewise definition of the field variable in the neighborhood of chosen control volumes. The total solution domain is divided into many small control volumes which are usually rectangular in shape.

- Boundary Conditions may be applied in finite volume equations by substituting for heat fluxes at boundaries or by changing the control volume spacing (while substituting for boundary temperatures).

- The finite volume method is a discretization of the governing equation in integral form, in contrast to the finite difference method, which is unusually applied to the governing equation in differential form.

- Finite volume method can be applied in first and second order equations and the discretized equation finally reduces to the central finite difference scheme on a uniform rectangular grid.

- One attractive feature of the finite volume method is that it can handle Neumann boundary condition as readily as the Dirichlet boundary condition.
REFERENCES

- Lecture notes of Professor G. Biswas
- Fletcher, 'Computational Techniques for Fluid Dynamics‘
- Peyret and Taylor, ‘Computational Methods for Fluid Flow'
Thank You!