LAMINAR, TRANSITIONAL AND TURBULENT FLOWS

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OUTLINE

- Introduction and classification of fluid flow
- Examples distinguishing three types of flows
- Laminar to turbulent transition
  1. Stable and Unstable Laminar Flow
  2. Orr-Sommerfeld Equation
  3. Transitional flows
  4. Rayleigh Equation
- Turbulent state and its analysis using Reynolds's equation
- A note on numerical methods available for solving fluid mechanics problems.
INTRODUCTION

- **Important parameter**: Reynolds number $\text{Re}$

- **Laminar Flow**: occurs when a fluid flows in parallel layers, with no disruption between the layers. No cross currents or eddies perpendicular to direction of flow.

- **Turbulent motion**: characterized by rapid mixing. Cross currents flow perpendicular to direction of motion.

- Above flow types can be further sub grouped as:
  - **Stable Laminar Flow**: These type of laminar flow prove stable towards imposed disturbances acting from outside.
  - **Unstable Laminar flow**: A laminar flow is considered unstable when disturbances introduced into it are amplified, but a certain "regularity" in the excited disturbance is maintained, i.e. due to the disturbance the investigated flow merges into a new laminar flow state.
- **Transitional Flow**: On applying external disturbance, we find there are irregular fluctuations. Intermittent laminar and turbulent flow occur, i.e. phases occur in the flow in which the flow is laminar and phases in which the flow shows turbulent characteristics.

- **Turbulent Flow**:
  1. characterized by high three-dimensionality
  2. Fluids motion is highly random and unpredictable.

![Fig 2.](image)

Above figures illustrate how on increasing the Reynolds number, we move from stable laminar region to unstable region. At low Re, the flow around cylinder possess symmetry but at higher Reynolds number, this symmetry doesn’t exist and two separate regions of different shape and length occur maintaining some sort of “regularity”.

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**Fig 3. Subsonic open jet with areas of laminar, transitional and turbulent flow**
LAMINAR TO TURBULENT TRANSITION

- The basic equation to start with is Navier Stokes Equation

Under all the assumptions, in 1-D case, this simplifies to

\[ \rho \frac{\partial U_1}{\partial t} = \mu \frac{\partial^2 U_1}{\partial x^2} \]  \hspace{1cm} (a)

Assuming initial flow velocity \( U_0 \) and superimposing on it a disturbance of amplitude \( u_A \), so total velocity at any time is

\[ U_1 = U_0 + u_A \sin(2\pi \frac{x}{\lambda}) \]

Substituting above equation into (a),
\[
\frac{du_A}{dt} = -\nu \frac{4\pi^2}{\lambda^2} u_A
\]

- Solving the equation with initial condition of \( u_A = (u_A)_0 \)

\[
u_A(t) = (u_A)_o \exp(-\nu \frac{4\pi^2}{\lambda^2} t)
\]

**Implications of the above equation**

1. This solution makes it clear that the viscosity terms in the momentum equations can be considered to lead to attenuations of imposed disturbances, i.e. disturbances which are introduced into a laminar flow field will be damped due to the viscosity of the fluid.

2. The resistance towards short-wave disturbances, i.e. disturbances with small \( \lambda \) values, turns out to be stronger, so that these receive stronger damping in the course of time. It is this attenuation effect, caused by the viscosity of a fluid, which ensures that many laminar flows possess high stability. This means that they show strong resistance against external disturbances.
Generally, gradients of flow and/or fluid properties can be stated as causes of amplification of disturbances. When they act on introduced disturbances such that an exponential excitation takes place-

\[ u_A(t) = (u_A)_o \exp(\alpha t) \]

When a viscosity-dependent attenuation exists at the same time, the net result that is obtained.

\[ u_A(t) = (u_A)_o \exp[(\alpha - \beta)t] \]

where \( \beta = \text{viscosity-dependent attenuation term} \)

If, \( \beta > \alpha \)- stable laminar flow

\( \alpha > \beta \)- unstable laminar flow
Orr-Sommerfeld Equation

This equation precisely determines the conditions for hydrodynamic stability. It is an eigenvalue equation describing a linear 2–dimensional mode of disturbance to a viscous parallel flow.

Consider a 1-D stationary velocity field $U_x(y)$. Now we introduce a two-dimensional time-dependent disturbance, such that we get the following total velocity fields

$$ u_1 = \hat{U}_x(y) + u_x(x, y, t) \quad u_2 = \hat{u}_y(x, y, t) \quad \hat{u}_3 = 0 \quad (1) $$

Furthermore, for pressure we assume that

$$ \hat{P} = P(x) + p'(x, y, t) \quad (2) $$

For all the fluctuations that we have introduced, we assume that

$$ \frac{u'_x}{U_x} \ll 1 \quad \frac{u'_y}{U_x} \ll 1 \quad \frac{p'}{P} \ll 1 \quad (3) $$
Substituting the expressions from (1), (2) and (3) in (4) and continuity equation

\[
\rho \left( \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right) = - \frac{\partial p}{\partial x} + \mu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) + \rho g_x \tag{4}
\]

\[
\frac{\partial u'_x}{\partial x} + \frac{\partial u'_y}{\partial y} = 0 \tag{5}
\]

\[
\frac{\partial u'_x}{\partial t} + (U'_x + u'_x) \frac{\partial u'_x}{\partial x} + u'_y \left( \frac{\partial U'_x}{\partial y} + \frac{\partial u'_x}{\partial y} \right) = - \frac{1}{\rho} \left( \frac{\partial p'}{\partial x} + \frac{\partial p'}{\partial x} \right) + \nu \left( \frac{\partial^2 u'_x}{\partial x^2} + \frac{\partial^2 u'_x}{\partial y^2} + \frac{\partial^2 U'_x}{\partial y^2} \right) \tag{6}
\]

\[
\frac{\partial u'_y}{\partial t} + (U'_x + u'_x) \frac{\partial u'_y}{\partial x} + u'_y \frac{\partial u'_y}{\partial x} = - \frac{1}{\rho} \frac{\partial p'}{\partial y} + \nu \left( \frac{\partial^2 u'_y}{\partial x^2} + \frac{\partial^2 u'_y}{\partial y^2} \right) \tag{7}
\]

Disregarding all squared terms in the disturbance and for a special case, assuming that

\[
0 = - \frac{1}{\rho} \frac{dP}{dx} + \nu \frac{d^2 U_x}{dy^2}
\]

we obtain
\[
\frac{\partial u_x'}{\partial x} + \frac{\partial u_y'}{\partial y} = 0 \quad (8)
\]

\[
\frac{\partial u_x'}{\partial t} + U_x \frac{\partial u_x'}{\partial x} + u_y' \frac{\partial U_x}{\partial y} = -\frac{1}{\rho} \frac{\partial p'}{\partial x} + \nu \left( \frac{\partial^2 u_x'}{\partial x^2} + \frac{\partial^2 u_x'}{\partial y^2} \right) \quad (9)
\]

\[
\frac{\partial u_y'}{\partial t} + U_x \frac{\partial u_y'}{\partial x} = -\frac{1}{\rho} \frac{\partial p'}{\partial y} + \nu \left( \frac{\partial^2 u_y'}{\partial x^2} + \frac{\partial^2 u_y'}{\partial y^2} \right) \quad (10)
\]

Introducing a stream function for the velocity field of the disturbances-

\[
u_x' = \frac{\partial \Psi'}{\partial y} \quad \text{and} \quad u_y' = -\frac{\partial \Psi'}{\partial x} \quad (11)
\]

Substituting these stream functions in equation in (9) and (10), we obtain following two equations

\[
\frac{\partial^2 \Psi'}{\partial y \partial t} + U_x \frac{\partial^2 \Psi'}{\partial x \partial y} - \frac{\partial \Psi'}{\partial x} \frac{dU_x}{dy} = -\frac{1}{\rho} \frac{\partial p'}{\partial x} + \nu \left( \frac{\partial^3 \Psi'}{\partial x^2 \partial y} + \frac{\partial^3 \Psi'}{\partial y^3} \right) \quad (12)
\]

\[
-\frac{\partial \Psi'}{\partial x \partial t} - U_x \frac{\partial^2 \Psi'}{\partial x^2} = -\frac{1}{\rho} \frac{\partial p'}{\partial y} - \nu \left( \frac{\partial^3 \Psi'}{\partial x^3} + \frac{\partial^3 \Psi'}{\partial x \partial y^2} \right) \quad (13)
\]
By differentiating (12) w.r.t \( y \) and (13) w.r.t \( x \), the pressure disturbance term can be eliminated, and we obtain a fourth order differential equation in terms of \( \Psi' \)

\[
\left( \frac{\partial}{\partial t} + U_x \frac{\partial}{\partial x} \right) \left( \frac{\partial^2 \Psi'}{\partial x^2} + \frac{\partial^2 \Psi'}{\partial y^2} \right) - \frac{d^2 U_x}{dy^2} \frac{\partial \Psi'}{\partial x} = \nu \left( -\frac{\partial^4 \Psi'}{\partial x^4} + 2\frac{\partial^4 \Psi'}{\partial x^2 \partial y^2} - \frac{\partial^4 \Psi'}{\partial y^4} \right) \tag{14}
\]

It is assumed that the following value of stream function satisfies equ. (14)

\[
\Psi' = f(y) \exp[i(kx - \omega t)] \tag{15}
\]

Where, \( k \) is a real quantity i.e. waves of wavelength \( \lambda = 2\pi/k \) are considered in the direction of the \( x \) coordinate, while, \( \omega \) can adopt complex values:

\[
\omega = \omega_R + i\omega_I \tag{16}
\]

- For \( \omega_I < 0 \), \( \Psi' \) decreases with time and the fluid flow \( U_x(y) \) can be regarded as stable.
- For \( \omega_I > 0 \), the disturbance is excited with time, i.e. the flow \( U_x(y) \), proves to be unstable with respect to the imposed disturbances.
Now to solve equation (14), we require implementation of four boundary conditions

- They can be stated for plane channel flows and flat plate boundary-layer flows as follows:
- For **Plane channel flows** at \( y = 0 \) and \( y = 2H \) we have \( u'_x = u'_y = 0 \), i.e.
  \[
  f(0) = f'(0) = f(2H) = f'(2H) = 0 
  \]  \hspace{1cm} (17)

- For **Flat plate boundary layer flows**, because of no slip condition at the wall, we obtain
  \[
  f(0) = f''(0) = 0
  \]

And then for the outer flow, we can state that due to lack of viscosity forces
\[
\frac{d^2 U_x}{dy^2} = 0,
\]
On setting $\Psi' = f(y) \exp[ik(x - ct)]$, we obtain the Orr-Sommerfeld differential equation

$$
(k U_x - \omega)(f'' - k^2 f) - k U_x' f = \frac{v}{ik} (f''' - 2k^2 f'' + k^4 f)
$$

(18)

This usually needs to be solved numerically for investigating the stability of a certain flow, using the undisturbed velocity distribution $U_x(y)$ and the assumed wavelength $k$ in the equation and employing the boundary conditions.

Here, $c = \omega/k = c_R + ic_I$. The wavelength range which leads to negative values of the imaginary part of $c$ are considered as stable, i.e. the investigated flow is stable with respect to the applied disturbances.

Thus, it is determined by successive computations, for which wavelength the imaginary part of $c$ is positive. This then leads to an insight into whether for a solution $U_x(y)$, that we have for a flow, the flow field $U_x(y)$ changes abruptly into a flow state differing from its undisturbed state.
Transitional Flows

- In a flow defined as laminar, the velocity signal shows a temporally constant velocity. In a flow defined as being turbulent, there exists, however, a time variation of the local velocity which shows velocity fluctuations around a mean value.
Transitional flow is a state of flow concerned with increase in disturbances with time as clear from previous slide. Transitional flow analysis is basically a subdomain of stability analysis carried out earlier, that its theory is based on Orr–Sommerfeld equation.

\[(kU_x - \omega)(f'' - k^2 f) - kU_x'' f = \frac{V}{ik} (f''' - 2k^2 f'' + k^4 f)\]  

(19)

For \(v=0\), above equation can be written as:

\[(kU_x - \omega)(f'' - k^2 f) - kU_x'' f = 0\]  

(20)

The above equation is Rayleigh Equation, requiring two boundary conditions which are as follows:

\[y = 0 \quad f(0) = 0 \quad \text{and} \quad y \to \infty \quad f(\to \infty) = 0\]  

(21)
From Rayleigh equation, we can draw the following inference

“All laminar velocity profiles which show an inflection point, i.e. for which at one location of the velocity profile $d^2U_1/dy^2 = 0$ holds, are unstable.”

It is also clear that curvature of velocity profile also has a very strong influence on the stability of the flow.
Instability diagram for Orr-Sommerfeld equation with and without viscosity

- This figure clearly shows that only a relatively narrow range of wavelengths and frequencies of disturbances have to be classified as “dangerous” for the stability of the boundary layer.

- When carrying out numerical computations, it results for $C = \omega/k$ that $C_R/U_\infty = 0.39$, $k\delta_1 = 0.36$ and $\omega_R\delta_1/U_\infty = 0.15$
- The smallest wavelength of the disturbances which can act in an unstable way on boundary layers is given as-
  \[ \lambda_{\text{min}} = (2\pi/0.36) * \delta_1 = 17.5\delta_1 = 6\delta \]
  For smaller wavelengths of disturbances, boundary layers prove to be stable.

- As the critical Reynolds number for the laminar-to-turbulent transition, numerical computations yield
  \[ \text{Re}_{\text{crit}} = 520 \]

- Corresponding experimentally obtained value:
  \[ (\text{Re}_{\text{crit}})^{\text{exp}} \approx 950. \]
  which is greater than the numerically obtained value.

- This difference is because numerical \( \text{Re}_{\text{crit}} = 520 \), represents the “point of neutral instability,” whereas the experimentally determined value probably represents the “point of the laminar-to-turbulent transition that occurs abruptly and these two points are not to be necessarily equal.
**Turbulent State**

- At very high Reynolds numbers, a flow state exists which stands out for its strong irregularity.

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**Re = 200**

(a) Time-dependent Laminar flow  
(b) Turbulent flow

Flow around a square cylinder in (a) Time-dependent Laminar flow  
(b) Turbulent flow
The analysis of turbulence state is quite different from that of laminar state as in turbulence state local instantaneous velocity is changing with time.

Here we define mean value of velocity as follows:

\[
\overline{U_j(x_i)} = \lim_{T \to \infty} \frac{1}{T} \int_0^T \hat{U_j}(x_i,t) dt \tag{1}
\]

This was first given by Reynolds to treat turbulent flows.
• Now introducing $u'_j(x_i,t)$, that is, turbulent velocity fluctuation, we define it in the following manner

$$u'_j(x_i,t) = U_j(x_i,t) - \overline{U_j(x_i)}$$

• Calculating time average of velocity of turbulent velocity fluctuations (2)

$$\overline{u_j(x_i,t)} = \lim_{T \to \infty} \frac{1}{T} \int_0^T u'_j(x_i,t) dt$$

(3)

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T u'_j(x_i,t) dt = \lim_{T \to \infty} \frac{1}{T} \int_0^T \hat{U}_j(x_i,t) dt - \lim_{T \to \infty} \frac{1}{T} \int_0^T \overline{U_j(x_i)} dt = 0$$

(4)

• The time average of the turbulent velocity fluctuations is equal to zero per definition. Hence, there is a way to present turbulence in local, time varying quantities, in a form such that the turbulent fluctuations of all flow quantities, that are introduced into the considerations, show a time mean value that is zero.
Just like velocity, we can write instantaneous values of pressure, temperature, and density in terms of their mean values and fluctuations.

Velocity

\[ \hat{U}_j(x_i, t) = u_j'(x_i, t) + \bar{U}_j(x_i) \]

Pressure

\[ \hat{P}(x_i, t) = p'(x_i, t) + \bar{P}(x_i) \]

Density

\[ \hat{\rho}(x_i, t) = \rho'(x_i, t) + \bar{\rho}(x_i) \]

Temperature

\[ \hat{T}(x_i, t) = t'(x_i, t) + \bar{T}(x_i) \]

Similarly, we can show that time average of these quantities is zero. That is

\[ u_j(x_i, t) = p(x_i, t) = \rho(x_i, t) = t(x_i, t) = 0 \quad (5) \]
Introducing the instantaneous quantities in the Navier–Stokes equations and, by time averaging the equations, a new set of equations results for the mean values of the flow properties, the so-called Reynolds equations.

Starting with continuity equation

\[ \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i} (\rho U_i) = 0 \]  

By replacing the instantaneous values of \( \rho \) and \( U \) with mean values and the corresponding turbulent fluctuations

\[ \frac{\partial}{\partial t} (\bar{\rho} + \rho') + \frac{\partial}{\partial x_i} \left[ \bar{\rho} + \rho' \right] \left[ \bar{U_i} + u_i' \right] = 0 \]  

By time averaging the above equation, we obtain

\[ \frac{\partial \bar{\rho}}{\partial t} + \frac{\partial}{\partial x_i} (\bar{\rho} \bar{U_i} + \rho' u_i') = 0 \]  

On subtracting (3) from (2), we get an expression for rate of change of instantaneous density.
\[
\frac{\partial \rho'}{\partial t} + \frac{\partial}{\partial x_i} (\rho u_i' + U_i \rho') = 0 \tag{4}
\]

For the fluids with constant density, we further simplify more, that is, \( \rho' = 0 \)

\[
\frac{\partial \bar{U}_i}{\partial x_i} = 0 \quad \text{and} \quad \frac{\partial u_i'}{\partial x_i} = 0 \tag{5}
\]

For fluids of constant density, the time-averaging of the continuity equation does not result in additional turbulent transport terms.

Now, beginning with Navier-Stokes equation

\[
\rho \left[ \frac{\partial U_j}{\partial t} + U_i \frac{\partial U_j}{\partial x_i} \right] = -\frac{\partial P}{\partial x_j} + \rho g_j + \frac{\partial}{\partial x_i} \left[ \nu \left( \frac{\partial U_j}{\partial x_i} + \frac{\partial U_i}{\partial x_j} \right) \right] \tag{6}
\]

As we are dealing with fluids of constant density, eq(5) holds true.

So,

\[
\frac{\partial^2 U_i}{\partial x_i \partial x_j} \frac{\partial}{\partial x_j} \left( \frac{\partial U_i}{\partial x_i} \right) = 0 \tag{7}
\]
Finally Navier Stokes equation can be written as

\[ \rho \left[ \frac{\partial U_j}{\partial t} + U_i \frac{\partial U_j}{\partial x_i} \right] = -\frac{\partial P}{\partial x_j} + \rho g_j + \nu \frac{\partial^2 U_j}{\partial x_i^2} \]  \hspace{1cm} (8)

Or taking help of equation (5), we can further write it as

\[ \rho \left[ \frac{\partial U_j}{\partial t} + \frac{\partial (U_j U_i)}{\partial x_i} \right] = -\frac{\partial P}{\partial x_j} + \rho g_j + \nu \frac{\partial^2 U_j}{\partial x_i^2} \]  \hspace{1cm} (9)

By replacing the instantaneous values of \( \rho \), \( P \) and \( U \) with summation of mean values and the corresponding turbulent fluctuations, we will obtain the following equation

\[ -\rho \left[ \frac{\partial (\overline{U}_j + u_j')}{\partial t} + \frac{\partial [(\overline{U}_j + u_j')(\overline{U}_i + u_i')]}{\partial x_i} \right] = 

- \frac{\partial (P + p')}{\partial x_j} + \rho g_j + \nu \frac{\partial^2 (\overline{U}_j + u_j')}{\partial x_i^2} \] \hspace{1cm} (10)
Time averaging of the previous equation, we will obtain

$$-\rho \left[ \frac{\partial \bar{U}_j}{\partial t} + \frac{\partial (\bar{U}_j \bar{U}_i + u_i' u_j')}{\partial x_i} \right] = -\frac{\partial \bar{P}}{\partial x_j} + \rho g_j + \nu \frac{\partial^2 \bar{U}_j}{\partial x_i^2} \quad (11)$$

Rearranging the terms we obtain the following equation

$$-\rho \frac{\partial}{\partial x_i} (\bar{U}_i \bar{U}_j) = -\frac{\partial \bar{P}}{\partial x_j} + \rho g_j + \frac{\partial}{\partial x_i} (\nu \frac{\partial \bar{U}_j}{\partial x_i} - \rho u_i' u_j') \quad (12)$$

If we look at above equation and compare it with Navier-Stokes equation. We see that additional terms are introduced into the Reynolds equations, which can be interpreted as additional momentum transport terms, so that for a turbulent fluid flow the following holds:

$$\tau_{ij}^{\text{tot}} = -\nu \frac{\partial \bar{U}_j}{\partial x_i} + \rho u_i' u_j' \quad (13)$$

The second term in the above equation is Reynolds stress tensor
So, it is clear that the splitting of the instantaneous velocity components into a mean component and a turbulent fluctuation leads to a division of the total momentum transport in a mean part and in a turbulent part. The total momentum transport develops on the one hand due to the mean flow field, $\rho U_i U_j$, and on the other due to correlations of the turbulent velocity fluctuations, $\rho u_i u_j$.

The introduction of additional unknowns into the basic equations of fluid mechanics lead to a non-closed system of equations and this requires additional information in order to obtain solutions from (5) and (12).

This additional information required is nothing but interrelation between Reynolds momentum transport terms and mean velocity terms.

A large number of experimental techniques have been employed to analyze turbulence: **hot wire and hot film anemometry**, measuring flow velocities indirectly. Then, there is **laser Doppler Technique** that measures velocity directly.
METHODS TO SOLVE FLUID MECHANICS PROBLEM

- Fluid motion is governed by the Navier-Stokes equations, a set of coupled and nonlinear partial differential equations derived from the basic laws of conservation of mass, momentum and energy. The unknowns are usually the velocity, the pressure and the density. The analytical paper and pencil solution of these equations is practically impossible save for the simplest of flows. Sometimes, solution is possible theoretically using Boundary Layer Theory.

- Scientists had to resort to laboratory experiments when theoretical analyses was impossible.

- The answers delivered are, however, usually qualitatively different since dynamical and geometric similitudes are difficult to enforce simultaneously between the lab experiment and the prototype. A prime example is the Reynolds’ number similarity which if violated can turn a turbulent flow laminar. Furthermore, the design and construction of these experiments can be difficult (and costly), particularly for stratified rotating flows.
Computational fluid dynamics (CFD) is an additional tool and quite an effective one. In its early days CFD was often controversial, as it involved additional approximation to the governing equations and raised additional (legitimate) issues.

Nowadays CFD is an established discipline alongside theoretical and experimental methods. This position is in large part due to the exponential growth of computer power which has allowed us to tackle ever larger and more complex problems.
The central process in CFD is the process of **discretization**, i.e. the process of taking differential equations with an *infinite* number of degrees of freedom, and reducing it to a system of *finite* degrees of freedom.

Hence, instead of determining the solution everywhere and for all times, we will be satisfied with its calculation at a finite number of locations and at specified time intervals. The partial differential equations are then reduced to a system of algebraic equations that can be solved on a computer.
Errors creep in during the discretization process. The nature and characteristics of the errors must be controlled in order to ensure that

- 1) we are solving the correct equations (consistency property), and
- 2) that the error can be decreased as we increase the number of degrees of freedom (stability and convergence).

Once these two criteria are established, the power of computing machines can be leveraged to solve the problem in a numerically reliable fashion.

Various discretization schemes have been developed to cope with a variety of issues. The most notable for our purposes are: finite difference methods, finite volume methods, finite element methods, and spectral methods.
CONCLUSIONS

- There is no single method such that we can solve all fluid mechanics problem using that particular method.
- The governing equations of flow also change as we from Laminar flow to turbulence state with addition of turbulent fluctuations which increases the complexities in the flow.
- Numerical methods seem to be most suitable ways of analyzing the fluid mechanics problem which give important insights to model turbulent flow.
- However, Numerical methods also become ineffective at very high Reynolds Number.
THANK YOU